An eigenvalue problem

Problem 86.2.4

Jan R. Magnus

Econometric Theory, 2, 290 (1986)

86.2.3. *Theil's minimum MSE estimator for the standard linear model*, proposed by Heinz Neudecker. Prove that Theil's "estimator" of $\boldsymbol{\beta}$ in the model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, $E(\boldsymbol{\varepsilon}) = 0$, $D(\boldsymbol{\varepsilon}) = \sigma^2 I$;

$$\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}'(\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}' + \sigma^2 I)^{-1}\mathbf{y}$$

systematically underestimates $\boldsymbol{\beta}$.

REFERENCE

H. Theil. *Principles of Econometrics*. (1971), p. 125.

86.2.4. *An Eigenvalue Problem*, proposed by Jan R. Magnus. Let **a** and **b** be two real $n \times 1$ vectors, and define the $n \times n$ matrices $\mathbf{A} = aa' + bb'$ and $\mathbf{B} = aa' - bb'$.

1. Find the eigenvalues of **A** and **B**.
2. Show that **B** is indefinite (that is, neither positive semidefinite nor negative semidefinite) unless a and b are linearly dependent.

Now, let **x** and **y** be two normally distributed $n \times 1$ vectors: $\mathbf{x} \cong N(\mu_1, \boldsymbol{\Omega})$, $\mathbf{y} \cong N(\mu_2, \boldsymbol{\Omega})$ with the same positive definite covariance matrix $\boldsymbol{\Omega}$. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ denote the covariance matrices of $\mathbf{x} \otimes \mathbf{x}$ and $\mathbf{y} \otimes \mathbf{y}$ respectively.

3. Show that $\mathbf{V}_1 - \mathbf{V}_2$ is indefinite unless μ_1 and μ_2 are linearly dependent.

86.2.5. *The distribution of LIML in the leading case*, proposed by R. W. Farebrother. Let

$$\mathbf{y}_1 = \mathbf{Y}_2\boldsymbol{\beta} + \mathbf{Z}_1\gamma + \mathbf{u}$$

be the first equation of a system of simultaneous equations with reduced form

$$\mathbf{Y} = \mathbf{Z}_1\boldsymbol{\Pi}_1 + \mathbf{Z}_2\boldsymbol{\Pi}_2 + \mathbf{V}$$

where $\mathbf{y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y} = [\mathbf{y}_1 \quad \mathbf{Y}_2]$ are $\mathbf{T} \times 1$, $\mathbf{T} \times n$ and $\mathbf{T} \times m$ matrices of observations on the endogenous variables, $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z} = [\mathbf{Z}_1 \quad \mathbf{Z}_2]$ are $\mathbf{T} \times \mathbf{K}_1$, $\mathbf{T} \times \mathbf{K}_2$ and $\mathbf{T} \times \mathbf{K}$ matrices of observations on the predetermined variables, **u** and **V** are $\mathbf{T} \times 1$ and $\mathbf{T} \times m$ matrices of disturbances, and $\boldsymbol{\beta}$, γ, $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ are $\mathbf{n} \times 1$, $\mathbf{K}_1 \times 1$, $\mathbf{K}_1 \times m$, and $\mathbf{K}_2 \times m$ matrices of parameters. The rows of **V** are independent random drawings from a multivariate normal distribution with mean zero and variance Ω.

Solution

Econometric Theory, 3, 467–469 (1987)

hence

$$\tilde{\beta} = \frac{1}{\sigma^2}\beta\beta'X'\left(I - \frac{1}{\sigma^2 + \beta'X'X\beta}X\beta\beta'X'\right)y,$$
$$= \frac{\beta'X'y}{\sigma^2 + \beta'X'X\beta}\beta.$$

Therefore,

$$E(\tilde{\beta}) = \frac{\beta'X'X\beta}{\sigma^2 + \beta'X'X\beta}\beta < \beta.$$

Solution 2 (Proposed by R.W. Farebrother): Let

$z = X\beta.$

Then

$(\sigma^2 + z'z)z' = z'(\sigma^{2\prime}I + zz'),$

so that

$z'(\sigma^2 I + zz')^{-1} = (\sigma^2 + z'z)^{-1}z',$

and Theil's "estimator"

$\tilde{\beta} = \beta z'(\sigma^2 I + zz')^{-1}y$

may be rewritten as

$\tilde{\beta} = \beta(\sigma^2 + z'z)^{-1}z'y.$

Now $E(y) = z$, so that

$E(\tilde{\beta}) = \beta z'z(\sigma^2 + z'z)^{-1}$

and Theil's estimator systematically shrinks β towards zero. (Another good solution has been suggested by Badi. H. Baltagi.)

86.2.4. *An Eigenvalue Problem* – Solution, proposed by Jan R. Magnus. The proof rests on the following lemma which can be proved using high-school algebra.

LEMMA. *The two equations in x_1 and x_2*

$x_1 + x_2 = p, \qquad x_1^2 + x_2^2 = q, \qquad (x_1, x_2 \in \mathbb{R}),$

with p and q in $\mathbb{R}$ having a solution (in $\mathbb{R}$) if and only if $p_2 \leq 2q$ in which case the solution is

$$x_{1,2} = \frac{1}{2}p \pm \frac{1}{2}\sqrt{2q - p^2}.$$

Moreover,

(1) *If* $0 < q < p^2$, *then* $x_{1,2}$ *are both positive or both negative.*
(2) *If* $0 \le p^2 < q$, *then one is positive and the other negative.*
(3) *If* $0 \le p^2 = q$, *then* $x_1 = 0$ *or* $x_2 = 0$ *or both.*

Proof of (1): The matrix A has eigenvalues λ_1, λ_2, and zero (multiplicity $n - 2$). Further,

$$a'a + b'b = \text{tr}A = \lambda_1 + \lambda_2,$$

$$(a'a)^2 + (b'b)^2 + 2(a'b)^2 = \text{tr}A^2 = \lambda_1^2 + \lambda_2^2.$$

Hence, using the Lemma,

$$\lambda_{1,2} = \frac{a'a + b'b}{2} \pm \frac{1}{2}\sqrt{(a'a - b'b)^2 + 4(a'b)^2}.$$

Similarly, B has eigenvalues μ_1, μ_2, and zero ($n - 2$ times) and

$$\mu_{1,2} = \frac{a'a - b'b}{2} \pm \frac{1}{2}\sqrt{(a + b)'(a + b)\cdot(a - b)'(a - b)}.$$

Proof of (2): We have $p \equiv \text{tr } B = a'a - b'b$ and

$q \equiv \text{tr } B^2 = (a'a)^2 + (b'b)^2 - 2(a'b)^2.$

Hence,

$$p^2 - q = -2((a'a)(b'b) - (a'b)^2) \le 0$$

with equality if and only if a and b are linearly dependent.

Proof of (3): From Magnus and Neudecker [1, Lemma 9], we have

$$V_1 = 2N(\Omega \otimes \Omega + \Omega \otimes \mu_1\mu_1' + \mu_1\mu_1' \otimes \Omega),$$

where $N = \frac{1}{2}(I + K)$ and K is the commutation matrix. Hence,

$$\begin{aligned} V_1 - V_2 &= 2N(\Omega \otimes (\mu_1\mu_1' - \mu_2\mu_2') + (\mu_1\mu_1' - \mu_2\mu_2') \otimes \Omega), \\ &= 4N(\Omega \otimes B)N, \end{aligned}$$

where $B = \mu_1\mu_1' - \mu_2\mu_2'$. If μ_1 and μ_2 are linearly dependent, then B is positive (or negative) semidefinite and thus $V_1 - V_2$ also. If μ_1 and μ_2 are linearly independent, then (2) implies that B can be written as

$$B = \lambda aa' - \mu bb'$$

with $\lambda > 0, \mu > 0, a'a = b'b = 1$, and $a'b = 0$. Let $S = I - aa'$ and $T = I - bb'$. Then

$$SBS = -\mu bb', \qquad TBT = \lambda aa',$$

so that

$$(\text{vec } S)'(V_1 - V_2)\text{vec } S = 4 \text{ tr } \Omega SBS = -\ 4\mu b'\Omega b < 0$$

and

$$(\text{vec } T)'(V_1 - V_2)\text{vec } T = 4 \text{ tr } \Omega TBT = 4\lambda a'\Omega a > 0.$$

REFERENCE

1. Magnus, J.R. & H. Neudecker. Symmetry, 0-1 matrices and Jacobians: A review. *Econometric Theory 2* (1986): 157-190.
2. Teräsvirta T. Superiority comparisons of homogeneous linear estimators. *Communications in Statistics: Theory and Methods.* A11 (1982), 1595-1601.

Note: Heinz Neudecker has noted to the Editor, after he sent the text to the publisher, that part of the questions of this problem have been answered by Timo Teräsvirta [2].

86.2.5. *The Distribution of LIML in the Leading Case*—Solution, proposed by Peter C.B. Phillips. Let $X = [\overset{1}{X_1}, \overset{n}{X_2'}]' \equiv N(0, I_m)$, where $m = n + 1$ and where we use the symbol "≡" to signify equality in distribution. The vector $h = X(X'X)^{-1/2}$ is distributed uniformly on the unit sphere in $\mathbb{R}^m$ and we may therefore write the LIML estimator in the form:

$$\tilde{\beta} \equiv -h_2/h_1 = -X_2/X_1 \equiv N(0, I_n)/N(0,1) \equiv \text{multivariate Cauchy.}$$

The final ≡ follows by elementary integration. Thus, setting $r = -X_2/X_1$ and $s = X_1^2$, we have

$$\text{pdf}(r) = (2\pi)^{-m/2} \int_0^\infty e^{-(1+r'r)s/2} s^{m/2-1} ds,$$

$$= \Gamma(m/2)\pi^{-m/2}(1 + r'r)^{-m/2}$$

as required.

86.2.6. *Moments of OLS and 2SLS Via Fractional Calculus*—Solution, proposed by John L. Knight.

$$\text{(i)} \qquad \psi(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(itz'z_1/z_1'z_1) \cdot \text{pdf}(z_1)\text{pdf}(z) dz_1 dz,$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[itz'\partial x/\Delta_x] e^{x'z_1} \text{pdf}(z_1) \cdot \text{pdf}(z) dz_1 dz \Bigg|_{x=0}.$$

Integrating with respect to z_1, we have

$$\int_{-\infty}^{\infty} e^{x'z_1} \text{pdf}(z_1) dz_1 = e^{x'\bar{z}_1 + x'x/2}$$

due to the normality of z_1.