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ON CERTAIN MOMENTS RELATING TO RATIOS OF QUADRATIC FORMS IN NORMAL VARIABLES : FURTHER RESULTS

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SUMMARY. Let y be a normally distributed $n \times 1$ vector with mean μ and covariance matrix (positive definite) Ω. Let $r = y'Ay/yBy$ be a ratio of quadratic forms where A is symmetric and B positive semi-definite (possible singular). We consider the moments $\theta_s = Er^s$, $\tau_s = Er^s(a'y)$, $\zeta_s = Ers(y'Cy)$, where a is an $n \times 1$ vector, C a symmetric $n \times n$ matrix (both nonrandom) and s a positive integer. The paper completely characterizes the existence of those moments (Theorems 1–3) and also provides exact expressions for them which can be calculated numerically, using one-dimensional integration routines. Several related results are also reported.

1. Motivation

Many estimators and test statistics in econometrics take the form of a ratio of two quadratic forms in normal variables. For example, the (double) k-class estimator in the simultaneous equations model, the least-squares estimator in the first-order autoregressive (AR(1)) model and many test statistics of restrictions on parameters in the linear model fall in this class. Perhaps the simplest example is provided by the AR(1) process $\{y_1, y_2, \ldots\}$ defined by

$$y_t = \beta y_{t-1} + u_t \ (t = 2, 3, \ldots)$$

where $|\beta| < 1$ and $\{u_1, u_2, \ldots\}$ is a sequence of i.i.d. $N(0, \sigma^2)$ random variables. The process $\{y_1, y_2, \ldots\}$ is a normal strictly stationary time series, provided we assume that the initial observation y_1 is specified by

$$y_1 = (1-\beta^2)^{-\frac{1}{2}} u_1.$$

Given a series of n observations $y = (y_1, y_2, \ldots, y_n)'$ generated by this stationary process, the least-squares estimator of β is given by

$$\hat{\beta} = \sum_{t=2}^{n} y_t y_{t-1} \Big/ \sum_{t=2}^{n} y_{t-1}^2.$$

This is a ratio of two quadratic forms, that is, we can write

$$\hat{\beta} = y'Ay/y'By \qquad \ldots \quad (1)$$

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for some symmetric $n \times n$ matrices A and B.

To obtain the moments of $\hat{\beta}$, we must have an exact expression for

$$\theta_s = E(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})^s. \qquad \dots \quad (2)$$

This was achieved in Magnus (1986) who also presented necessary and sufficient conditions for the *existence* of θ_s. See also Smith (1988) for a different approach using zonal polynomials and invariant polynomials with multiple matrix arguments.

If an estimator of the type (1) is used for forecasting and we wish to evaluate the moments of the (multiperiod) forecast error, then we need more than θ_s. In the simple example of the AR(1) model presented above, the s periods ahead forecast is

$$\hat{y}_{n+s} = \hat{\beta}^s y_n \ (s = 1, 2, \dots)$$

and the forecast error is therefore

$$\hat{y}_{n+s} - y_{n+s} = (\hat{\beta}^s - \beta^s) y_n - \sum_{j=0}^{s-1} \beta^j u_{n+s-j}.$$

Hence the forecast bias is

$$E(\hat{y}_{n+s} - y_{n+s}) = E^s \hat{\beta}^s y_n$$

and the mean-square forecast error is

$$E(\hat{y}_{n+s} - y_{n+s})^2 = \beta^{2s} E y_n^2 + E(\hat{\beta}^{2s} y_n^2) - 2\beta^s E(\hat{\beta}^s y_n^2) + \sigma^2 \sum_{j=0}^{s-1} \beta^{2j},$$

provided the expectations exist. In this particular case the forecast bias is zero, but in more general cases this will no longer be true. (For example, in the AR(1) model with an intercept the forecast bias vanishes only if the process is mean stationary ; see Magnus and Pesaran, 1987). Hence in order to obtain the forecast bias and mean-square forecast error of estimators of the type (1) we must have exact expressions for

$$\tau_s = E(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})^s(\boldsymbol{a}'\boldsymbol{y}) \qquad \dots \quad (3)$$

and

$$\zeta_s = E(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})^s(\boldsymbol{y}'\boldsymbol{C}\boldsymbol{y}) \qquad \dots \quad (4)$$

for an arbitrary $n \times 1$ vector $\boldsymbol{a}$ and symmetric $n \times n$ matrix $\boldsymbol{C}$, and obtain necessary and sufficient conditions for the existence of these expectations.

The purpose of the present paper is to derive such expressions and obtain such conditions. In Section 2 we study the existence of τ_s and ζ_s (and also of θ_s, since the result given in Magnus (1986), though correct, is unnecessarily complicated). In Section 3 we obtain the first and second derivatives (with respect to $\boldsymbol{\mu}$) of θ_s when $y \sim N(\boldsymbol{\mu}, \boldsymbol{\Omega})$, and in Section 4 we present exact expressions for τ_s and ζ_s. Section 5 contains some remarks on computational aspects. All proofs are contained in the Appendix.

The theoretical results presented here have proved useful in evaluating the exact forecast bias and mean-square forecast error of the AR(1) model (with and without intercept, stationary and nonstationary), see Hoque, Magnus and Pesaran (1988) and Magnus and Pesaran (1987, 1989) but they should prove equally useful in many other cases where estimators of the type (1) are involved.

2. The existence of θ_s, τ_s and ζ_s

Before we derive exact formulae for the expectations defined by τ_s and ζ_s (an exact expression for θ_s was obtained by Magnus, 1986), we shall address the problem of their existence.

An important result concerning the existence of moments of estimators was obtained by Kinal (1980, Proposition 1) who studied the k-class estimator. The following result, from which his proposition is an immediate consequence, is in essence also due to Kinal (1980, Theorem 1) :

Kinal's Lemma : *Let* $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ *be independently distributed* $k \times 1$ *vectors such that* $\boldsymbol{x}_i \sim N(\mu_i, \sigma_i^2 I_k)$ *for* $i = 1, \ldots, n$. *Let* $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)'$ *and assume* $n \geqslant k$. *Then*

(a) $|\boldsymbol{X}'\boldsymbol{X}| > 0$ *with probability* 1 ;

(b) *the s-th moment of* $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ *exists if and only if* $s < (n-k+1)/2$;

(c) *the s-th moment of* $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ *exists if and only if* $s < n-k+1$.

In order to prove the existence results stated below as Theorems 1, 2, and 3, we shall need the special case of Kinal's Lemma where $k = 1$. The setup of all three theorems is given by Assumption 1.

Assumption 1 : Let $\boldsymbol{y}$ be a normally distributed $n \times 1$ vector with mean $\boldsymbol{\mu}$ and positive definite covariance matrix $\boldsymbol{\Omega}$. Let $\boldsymbol{A}$ be a symmetric $n \times n$ matrix and $\boldsymbol{B}$ a positive semidefinite $n \times n$ matix of rank $r \geqslant 1$. If $r \leqslant n-1$, let $\boldsymbol{Q}$ be an $n \times (n-r)$ matrix of full column-rank $n-r$ such that $\boldsymbol{BQ} = \boldsymbol{0}$.

It will become clear from the formulation of Theorems 1—3 that the structure of $\boldsymbol{\Omega}$ is irrelevant as far as the existence of θ_s, τ_s and ζ_s is concerned (assuming, of course, that $\boldsymbol{\Omega}$ is nonsingular). Thus Theorem 7 of Magnus (1986) which considers the existence of θ_s, while correct, is not as general as it should be. The general result follows.

Theorem 1 : *Let Assumption* 1 *hold and consider, for* $s \geqslant 0$,

$$\theta_s = E(\boldsymbol{y}'\boldsymbol{Ay}/\boldsymbol{y}'\boldsymbol{By})^s,$$

(i) *If* $r \leqslant n-1$ *and* $\boldsymbol{AQ} = \mathbf{0}$, *or if* $r = n$, *then* θ_s *exists for all* $s \geqslant 0$;

(ii) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} = \mathbf{0}$ *and* $\boldsymbol{AQ} \neq \mathbf{0}$, *then* θ_s *exists for* $0 \leqslant s < r$ *and does not exist for* $s \geqslant r$; *and*

(iii) *if* $r \leqslant n-1$ *and* $\boldsymbol{Q'AQ} \neq 0$, *then* θ_s *exists for* $0 \leqslant s < r/2$ *and does not exist for* $s \geqslant r/2$.

The corresponding existence conditions for τ_s and ζ_s are given in Theorems 2 and 3.

Theorem 2 : *Let Assumption* 1 *hold, let* $\boldsymbol{a}$ *be an* $n \times 1$ *vector and consider, for* $s \geqslant 0$,

$$\tau_s = E(\boldsymbol{y'Ay}/\boldsymbol{y'By})^s\,(\boldsymbol{a'y}).$$

(i) *If* $r \leqslant n-1$ *and* $\boldsymbol{AQ} = \mathbf{0}$, *or if* $r = n$, *then* τ_s *exists for all* $s \geqslant 0$;

(ii) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} = \mathbf{0}$, $\boldsymbol{AQ} \neq \mathbf{0}$ *and* $\boldsymbol{Q'a} = \mathbf{0}$, *then* τ_s *exists for* $0 \leqslant s < r+1$ *and does not exist for* $s \geqslant r+1$;

(iii) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} = \mathbf{0}$, $\boldsymbol{AQ} \neq \mathbf{0}$, *and* $\boldsymbol{Q'a} \neq \mathbf{0}$, *then* τ_s *exists for* $0 \leqslant s < r$ *and does not exist for* $s \geqslant r$;

(iv) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} \neq \mathbf{0}$ *and* $\boldsymbol{Q'a} = \mathbf{0}$, *then* τ_s *exists for* $0 \leqslant s < (r+1)/2$ *and does not exist for* $s \geqslant (r+1)/2$; *and*

(v) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} \neq 0$ *and* $\boldsymbol{Q'a} \neq \mathbf{0}$, *then* τ_s *exists for* $0 \leqslant s < r/2$ *and does not exist for* $s \geqslant r/2$.

Theorem 3 : *Let Assumption* 1 *hold, let* $\boldsymbol{C}$ *be a symmetric* $n \times n$ *matrix and consider, for* $s \geqslant 0$,

$$\zeta_s = E(\boldsymbol{y'Ay}/\boldsymbol{y'By})^s\,(\boldsymbol{y'Cy}).$$

(i) *If* $r \leqslant n-1$ *and* $\boldsymbol{AQ} = \mathbf{0}$, *or if* $r = n$, *then* ζ_s *exists for all* $s \geqslant 0$;

(ii) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} = \mathbf{0}$, $\boldsymbol{AQ} \neq \mathbf{0}$ *and* $\boldsymbol{CQ} = \mathbf{0}$, *then* ζ_s *exists for* $0 \leqslant s < r+2$ *and does not exist for* $s \geqslant r+2$;

(iii) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} = \mathbf{0}$, $\boldsymbol{AQ} \neq \mathbf{0}$, $\boldsymbol{Q'CQ} = \mathbf{0}$ *and* $\boldsymbol{CQ} \neq \mathbf{0}$, *then* ζ_s *exists for* $0 \leqslant s < r+1$ *and does not exist for* $s \geqslant r+1$;

(iv) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} = \mathbf{0}$, $\boldsymbol{AQ} \neq 0$ *and* $\boldsymbol{Q'CQ} \neq \mathbf{0}$, *then* ζ_s *exists for* $0 \leqslant s < r$ *and does not exist for* $s \geqslant r$;

(v) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} \neq \mathbf{0}$ *and* $\boldsymbol{CQ} = \mathbf{0}$, *then* ζ_s *exists for* $0 \leqslant s < (r+2)/2$ *and does not exist for* $s \geqslant (r+2)/2$;

(vi) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} \neq \mathbf{0}$, $\boldsymbol{Q'CQ} = \mathbf{0}$ *and* $\boldsymbol{CQ} \neq \mathbf{0}$, *then* ζ_s *exists for* $0 \leqslant s < (r+1)/2$ *and does not exist for* $s \geqslant (r+1)/2$; *and*

(vii) *if* $r \leqslant n-1$, $\boldsymbol{Q'AQ} \neq \mathbf{0}$ *and* $\boldsymbol{Q'CQ} \neq \mathbf{0}$, *then* ζ_s *exists for* $0 \leqslant s < r/2$ *and does not exist for* $s \geqslant r/2$.

Proofs of all the theorems are in the Appendix.

3. THE DERIVATIVES OF θ_s

In this section we shall find expressions for the first and second derivatives (with respect to μ) of $E(y'Ay/y'By)^s$ at an arbitrary point μ in $\mathcal{R}^n$ and for an arbitrary positive integer s when $y \sim N(\mu, \Omega)$. These derivatives are required in the proof of Theorem 5 and are of interest in themselves. Hoque (1985, Appendix B) obtained the special cases $s = 1$ and $s = 2$ for arbitrary μ, while Hoque, Magnus and Pesaran (1988, Proposition 2) proved the special case $\mu = 0$ for arbitrary s.

Theorem 4. *Let y_0 be a normally distributed $n \times 1$ vector with mean μ_0 and positive definite covariance matrix $\Omega_0 = LL'$. Let A be a symmetric $n \times n$ matrix and B a positive semidefinite $n \times n$ matrix, $B \neq 0$. For every μ in $\mathcal{R}^n$ define*

$$y = y(\mu) = y_0 + \mu - \mu_0,$$

so that $y(\mu_0) = y_0$ and $y(\mu) \sim N(\mu, \Omega_0)$. Let s be a positive integer and g_s a real-valued function defined by

$$g_s(x) = \begin{cases} (x'Ax/x'Bx)^s & \text{if } x \in \mathcal{R}^n - \mathcal{N}(B) \\ 0 & \text{if } x \in \mathcal{N}(B), \end{cases}$$

where $\mathcal{N}(B)$ denotes the nullspace of B, i.e. the set $\{x \in \mathcal{R}^n, Bx = 0\}$. If, for every μ sufficiently close to μ_0, $g_s(y(\mu))$ is a random variable with finite expectation, say $\theta_s(\mu)$, then

(a) *θ_s is ∞ times continuously differentiable at $\mu = \mu_0$;*

(b) *the first derivative of θ_s at $\mu = \mu_0$ is*

$$\left.\frac{\partial \theta_s(\mu)}{\partial \mu}\right|_{\mu=\mu_0} = \left\{\frac{1}{(s-1)!} \sum_{v} \gamma_s(v) \int_0^\infty t^{s-1} |\Delta| e^{-\frac{1}{2}\mu_0' V \mu_0} (2W - \omega V) dt\right\} \mu_0;$$

(c) *the second derivative of θ_s at $\mu = \mu_0$ is*

$$\left.\frac{\partial^2 \theta_s(\mu)}{\partial \mu \partial \mu'}\right|_{\mu=\mu_0} = \frac{1}{(s-1)!} \sum_{v} \gamma_s(v) \int_0^\infty t^{s-1} |\Delta| e^{-\frac{1}{2}\mu_0' V \mu_0} Z_s(t) dt.$$

Here

$$\begin{aligned} Z_s(t) = {} & 2W - \omega(V - V\mu_0\mu_0' V) - 2V\mu_0\mu_0' W - 2W\mu_0\mu_0' V \\ & + 4 \sum_{j=1}^{s} j^2 n_j (n_j - 1) r_j^* \omega_j W_j \mu_0 \mu_0' W_j \\ & + 8 \sum_{i<j} \omega_{ij} (i n_i r_i W_i) \mu_0 \mu_0' (j n_j r_j W_j), \end{aligned}$$

P is an orthogonal $n \times n$ matrix and Λ a diagonal $n \times n$ matrix such that

$$P'L'BLP = \Lambda, \; P'P = I_n,$$

the summations in (b) and (c) are over all $1\times s$ vectors $\nu=(n_1, n_2, \ldots, n_s)$ whose elements n_j are nonnegative integers satisfying $n_1+2n_2+\ldots+sn_s=s$;

$$\gamma_s(\nu)=s\,!\,2^s\prod_{j=1}^{s}\{n_j\,!\,(2j)^{n_j}\}^{-1},$$

$$\boldsymbol{\Delta}=(\boldsymbol{I}_n+2t\boldsymbol{\Lambda})^{-\frac{1}{2}},\ \boldsymbol{R}=\boldsymbol{\Delta P'L'ALP\Delta},$$

$$\boldsymbol{V}=\boldsymbol{\Omega}_0^{-1}-\boldsymbol{L}'^{-1}\boldsymbol{P\Delta}^2\boldsymbol{P}'\boldsymbol{L}^{-1},\ \boldsymbol{W}_j=\boldsymbol{L}'^{-1}\,\boldsymbol{P\Delta R}^j\,\boldsymbol{\Delta P}'\boldsymbol{L}^{-1},$$

$$\omega=\prod_{j=1}^{s}\varphi_j^{n_j},\ \varphi_j=\operatorname{tr}\boldsymbol{R}^j+j\boldsymbol{\mu}_0'\boldsymbol{W}_j\boldsymbol{\mu}_0,$$

$$\omega_j=\begin{cases}1 & \text{if } s=1\\ \prod\limits_{\substack{i=j\\ i\neq j}}^{s}\varphi_i^{n_i} & \text{if } s\geqslant 2,\end{cases}$$

$$\omega_{ij}=\begin{cases}1 & \text{if } s=1\\ \prod\limits_{\substack{h=1\\ h\notin\{i,j\}}}^{s}\varphi_h^{n_h} & \text{if } s\geqslant 2,\end{cases}$$

$$r_j=\begin{cases}1 & \text{if } n_j=0,1\\ \varphi_j^{n_j-1} & \text{if } n_j\geqslant 2,\end{cases}$$

$$r_j^*=\begin{cases}1 & \text{if } n_j=0,1,2\\ \varphi_j^{n_j-2} & \text{if } n_j\geqslant 3,\end{cases}$$

and

$$\boldsymbol{W}=\sum_{j=1}^{s}jn_jr_j\omega_j\boldsymbol{W}_j.$$

4. An exact expression for τ_s and ζ_s

We are now ready to state and prove our main result. If the conditions for the existence of τ_s, and ζ_s, given in Theorems 2 and 3 are satisfied, we shall obtain exact formulae for these expectations. Application of Theorem 5 does *not* require s-fold differentiation (as in Sawa (1972), see also Magnus (1986, Lemma 4)), but it does require numerical evaluation of a univariate integral ; see Magnus (1986, Section 7) and Section 5 below for some remarks on the computation of such integrals. That Theorem 5 is practicable is shown by the exact results obtained in Hoque, Magnus and Pesaran (1988) and Magnus and Pesaran (1987, 1989).

Theorem 5. *Let* $\boldsymbol{y}$ *be a normally distributed* $n\times 1$ *vector with mean* $\boldsymbol{\mu}$ *and positive definite covariance matrix* $\boldsymbol{\Omega}=\boldsymbol{L}\boldsymbol{L}'$. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{C}$ *be symmetric* $n\times n$ *matrices,* $\boldsymbol{B}$ *a positive semidefinte* $n\times n$ *matrix,* $\boldsymbol{B}\neq\boldsymbol{0}$, *and* $\boldsymbol{a}$ *an* $n\times 1$ *vector. Let* $\boldsymbol{P}$ *be an orthogonal* $n\times n$ *matrix and* $\boldsymbol{\Lambda}$ *a diagonal* $n\times n$ *matrix such that*

$$\boldsymbol{P}'\boldsymbol{L}'\boldsymbol{B}\boldsymbol{L}\boldsymbol{P}=\boldsymbol{\Lambda},\ \boldsymbol{P}'\boldsymbol{P}=\boldsymbol{I}_n$$

and define

$$\boldsymbol{A}^*=\boldsymbol{P}'\boldsymbol{L}'\boldsymbol{A}\boldsymbol{L}\boldsymbol{P},\ \boldsymbol{C}^*=\boldsymbol{P}'\boldsymbol{L}'\boldsymbol{C}\boldsymbol{L}\boldsymbol{P},\ \boldsymbol{a}^*=\boldsymbol{P}'\boldsymbol{L}'\boldsymbol{a},\ \boldsymbol{\mu}^*=\boldsymbol{P}'\boldsymbol{L}^{-1}\boldsymbol{\mu}.$$

Then we have, provided the expectations exist, for $s=1,2,\dots,$

$$\text{(a)}\quad E\{(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})^s(\boldsymbol{a}'\boldsymbol{y})\}=\frac{\exp\left(-\tfrac{1}{2}\boldsymbol{\mu}^{*\prime}\boldsymbol{\mu}^*\right)}{(s-1)\,!}\sum_{\nu}\gamma_s(\nu)\int_0^\infty t^{s-1}|\Delta|\,e^{\frac{1}{2}\boldsymbol{\xi}'\boldsymbol{\xi}}f_s(t)dt\,;$$

$$\text{(b)}\quad E\{(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})^s\,(\boldsymbol{y}'\boldsymbol{C}\boldsymbol{y})\}=\frac{\exp\left(-\tfrac{1}{2}\boldsymbol{\mu}^{*\prime}\boldsymbol{\mu}^*\right)}{(s-1)\,!}\sum_{\nu}\gamma_s(\nu)\int_0^\infty t^{s-1}|\Delta|\,e^{\frac{1}{2}\boldsymbol{\xi}'\boldsymbol{\xi}}g_s(t)dt.$$

Here

$$f_s(t)=\omega(\boldsymbol{d}'\boldsymbol{\xi})+2\sum_{j=1}^{s}jn_jr_j\,\omega_j(\boldsymbol{d}'\boldsymbol{R}^j\boldsymbol{\xi}),$$

$$\begin{aligned}g_s(t)=\;&\omega(\operatorname{tr}\boldsymbol{\Gamma}+\boldsymbol{\xi}'\boldsymbol{\Gamma}\boldsymbol{\xi})\\&+2\sum_{j=1}^{s}jn_jr_j\,\omega_j(\operatorname{tr}\boldsymbol{R}^j\boldsymbol{\Gamma}+2\boldsymbol{\xi}'\boldsymbol{R}^j\boldsymbol{\Gamma}\boldsymbol{\xi})\\&+4\sum_{j=1}^{s}j^2\,n_j(n_j-1)r_j^*\;\omega_j\,(\boldsymbol{\xi}'\boldsymbol{R}^j\boldsymbol{\Gamma}\boldsymbol{R}^j\boldsymbol{\xi})\\&+8\sum_{i<j}\omega_{ij}(in_i\,r_i)(jn_j\,r_j)(\boldsymbol{\xi}'\boldsymbol{R}^i\boldsymbol{\Gamma}\boldsymbol{R}^j\boldsymbol{\xi}),\end{aligned}$$

the summations in (a) and (b) are over all $1\times s$ vectors $\nu=(n_1,n_2,\dots,n_s)$ whose elements n_j are nonnegative integers satisfying $\sum_{j=1}^{s}n_j=s$,

$$\gamma_s(\nu)=s\,!\;2^s\prod_{j=1}^{s}\{n_j\,!\;(2j)^{n_j}\}^{-1},$$

$\boldsymbol{\Delta}$ is a diagonal positive definite $n\times n$ matrix, $\boldsymbol{R}$ and $\boldsymbol{\Gamma}$ symmetric $n\times n$ matrices and $\boldsymbol{d}$ and $\boldsymbol{\xi}$ are $n\times 1$ vectors defined as

$$\boldsymbol{\Delta}=(\boldsymbol{I}_n+2t\boldsymbol{\Lambda})^{-\frac{1}{2}},\ \boldsymbol{R}=\boldsymbol{\Delta}\boldsymbol{A}^*\boldsymbol{\Delta},\ \boldsymbol{\Gamma}=\boldsymbol{\Delta}\boldsymbol{C}^*\boldsymbol{\Delta},\ \boldsymbol{d}=\boldsymbol{\Delta}\boldsymbol{a}^*,\ \boldsymbol{\xi}=\boldsymbol{\Delta}\boldsymbol{\mu}^*,$$

and the scalars ω, ω_j, ω_{ij}, r_j and r_j^* are defined by

$$\omega = \prod_{j=1}^{s} \varphi_j^{n_j}, \quad \varphi_j = \operatorname{tr} \boldsymbol{R}^j + j\boldsymbol{\xi}'\boldsymbol{R}^j\boldsymbol{\xi},$$

$$\omega_j = \begin{cases} 1 & \text{if } s = 1 \\ \prod_{\substack{i=1 \\ i \neq j}}^{s} \varphi_i^{n_i} & \text{if } s \geqslant 2, \end{cases}$$

$$\omega_{ij} = \begin{cases} 1 & \text{if } s = 1 \\ \prod_{\substack{h=1 \\ h \neq \{i,j\}}}^{s} \varphi_h^{n_h} & \text{if } s \geqslant 2, \end{cases}$$

$$r_j = \begin{cases} 1 & \text{if } n_j = 0,\ 1 \\ \varphi_j^{n_j-1} & \text{if } n_j \geqslant 2, \end{cases}$$

$$r_j^* = \begin{cases} 1 & \text{if } n_j = 0,\ 1,\ 2 \\ \varphi_j^{n_j-2} & \text{if } n_j \geqslant 3. \end{cases}$$

5. Some remarks on computation

It is easily seen from Magnus (1986, Theorem 6) and from Theorem 5 that θ_s, τ_s and ζ_s can all three be expressed as

$$\frac{\exp\left(-\frac{1}{2}\boldsymbol{\mu}^{*\prime}\boldsymbol{\mu}^*\right)}{(s-1)!} \int_0^\infty t^{s-1} |\boldsymbol{\Delta}|\, e^{\frac{1}{2}\boldsymbol{\xi}'\boldsymbol{\xi}} \left[\sum_\nu \gamma_s(\nu) h_s(\nu, t)\right] dt, \qquad \ldots \quad (5)$$

where $h_s(\nu, t)$ is different in the three cases but the rest in expression (5) is the same. The integral in (5) has to be evaluated numerically. As pointed out in Magnus (1986), the entire integration range $(0, \infty)$ can first be transformed to $(0, 1)$ using the identity

$$\int_0^\infty f(t)dt = \int_0^1 f\left(\frac{1-z}{z}\right) \frac{1}{z^2}\, dz.$$

Numerical evaluation of the univariate integral causes no particular difficulties unless n is large or Ω is close to being singular. The Numerical Algorithms Group (1984) (the so-called NAG), among others, contains subroutines that can be used for this purpose. Subroutine DO1AMF does the numerical integration and also gives an estimate of the absolute error in the integration (which I usually set at 10^{-5}). Subroutine F02ABF calculates the eigenvalues and the eigenvectors in $\boldsymbol{\Lambda}$ and P.

Appendix

Proof of Theorems 1, 2 *and* 3 : If $r = n$, then $\boldsymbol{B}$ is positive definite, $\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y}$ is bounded by the extreme eigenvalues of $\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}}$, and hence θ_s, τ_s and ζ_s exist for all $s \geqslant 0$.

Assume next that $1 \leqslant r \leqslant n-1$. Let $\boldsymbol{L}$ be a nonsingular $n \times n$ matrix such that $\boldsymbol{\Omega} = \boldsymbol{L}\boldsymbol{L}'$ and let $\boldsymbol{P}_1$ be an $n \times r$ matrix satisfying

$$\boldsymbol{L}'\boldsymbol{B}\boldsymbol{L}\boldsymbol{P}_1 = \boldsymbol{P}_1\boldsymbol{\Lambda}_1,\ \boldsymbol{P}_1'\boldsymbol{P}_1 = \boldsymbol{I}_r,\ \boldsymbol{P}_1'\boldsymbol{L}^{-1}\boldsymbol{Q} = \boldsymbol{0},$$

where $\boldsymbol{\Lambda}_1$ is a diagonal $r \times r$ matrix containing the r positive eigenvalues of $\boldsymbol{L}'\boldsymbol{B}\boldsymbol{L}$. Let $\boldsymbol{P}_2 = \boldsymbol{L}^{-1}\boldsymbol{Q}(\boldsymbol{Q}'\boldsymbol{\Omega}^{-1}\boldsymbol{Q})^{-\frac{1}{2}}$ and define

$$\boldsymbol{x}_1 = \boldsymbol{P}_1'\boldsymbol{L}^{-1}\boldsymbol{y}, \quad \boldsymbol{x}_2 = \boldsymbol{P}_2'\boldsymbol{L}^{-1}\boldsymbol{y}.$$

Then $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are independently normally distributed, and

$$\boldsymbol{P}_1\boldsymbol{P}_1' + \boldsymbol{P}_2\boldsymbol{P}_2' = \boldsymbol{I}_n.$$

Letting

$$\boldsymbol{D}_{11} = \boldsymbol{P}_1'\boldsymbol{L}'\boldsymbol{A}\boldsymbol{L}\boldsymbol{P}_1,\ \boldsymbol{D}_{12} = \boldsymbol{P}_1'\boldsymbol{L}'\boldsymbol{A}\boldsymbol{L}\boldsymbol{P}_2,\ \boldsymbol{D}_{22} = \boldsymbol{P}_2'\boldsymbol{L}'\boldsymbol{A}\boldsymbol{L}\boldsymbol{P}_2,$$

we obtain

$$\frac{\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y}} = \frac{\boldsymbol{x}_1'\boldsymbol{D}_{11}\boldsymbol{x}_1 + 2\boldsymbol{x}_1'\boldsymbol{D}_{12}\boldsymbol{x}_2 + \boldsymbol{x}_2'\boldsymbol{D}_{22}\boldsymbol{x}_2}{\boldsymbol{x}_1'\boldsymbol{\Lambda}_1\boldsymbol{x}_1}.$$

Now, let $\varphi^*(\boldsymbol{y})$ be a random variable all whose moments exist. Then, writing

$$\varphi^*(\boldsymbol{y}) = \varphi^*(\boldsymbol{L}\boldsymbol{P}_1\boldsymbol{x}_1 + \boldsymbol{L}\boldsymbol{P}_2\boldsymbol{x}_2) \equiv \varphi(\boldsymbol{x}_1, \boldsymbol{x}_2),$$

we see that $E(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})^s\varphi^*(\boldsymbol{y})$ exists if and only if

$$E(\boldsymbol{x}_1'\boldsymbol{D}_{12}\boldsymbol{x}_2/\boldsymbol{x}_1'\boldsymbol{x}_1)^s\,\varphi(\boldsymbol{x}_1, \boldsymbol{x}_2)$$

and

$$E(\boldsymbol{x}_2'\boldsymbol{D}_{22}\boldsymbol{x}_2/\boldsymbol{x}_1'\boldsymbol{x}_1)^s\,\varphi(\boldsymbol{x}_1, \boldsymbol{x}_2)$$

both exist.

As a special case of Kinal's Lemma (see section 2) we find that

$$E(1/\boldsymbol{x}_1'\boldsymbol{x}_1)^s \text{ exists if and only if } s < r/2 \qquad \ldots \quad (\text{A.1})$$

and

$$E(\boldsymbol{x}_1'\boldsymbol{z}/\boldsymbol{x}_1'\boldsymbol{x}_1)^s \text{ exists if and only if } s < r, \qquad \ldots \quad (\text{A.2})$$

when $\boldsymbol{z}$ is distributed independently of $\boldsymbol{x}_1$ ($\boldsymbol{z}$ may be nonrandom). Using (A.1) and (A.2) and the independence of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ we can then conclude the proof as follows.

In Theorem 1 we have $\varphi^*(\boldsymbol{y}) = \varphi(\boldsymbol{x}_1, \boldsymbol{x}_2) = 1$ and we distinguish, as in the theorem, between three cases :

(i) $\boldsymbol{D}_{22} = \boldsymbol{0}$, $\boldsymbol{D}_{12} = \boldsymbol{0}$. Then θ_s exists for all $s \geqslant 0$;

(ii) $\boldsymbol{D}_{22} = \boldsymbol{0}$, $\boldsymbol{D}_{12} \neq \boldsymbol{0}$. θ_s exists $\Longleftrightarrow E(\boldsymbol{x}_1'\boldsymbol{D}_{12}\boldsymbol{x}_2/\boldsymbol{x}_1'\boldsymbol{x}_1)^s$ exists $\Longleftrightarrow s < r$;

(iii) $\boldsymbol{D}_{22} \neq \boldsymbol{0}$. θ_s exists $\Longleftrightarrow E(1/\boldsymbol{x}_1'\boldsymbol{x}_1)^s$ exists $\Longleftrightarrow s < r/2$.

In Theorem 2 we define

$$\boldsymbol{d}_1 = \boldsymbol{P}_1'\boldsymbol{L}'\boldsymbol{a} \text{ and } \boldsymbol{d}_2 = \boldsymbol{P}_2'\boldsymbol{L}'\boldsymbol{a},$$

that

$$\varphi^*(\boldsymbol{y}) = \boldsymbol{a}'\boldsymbol{y} = \boldsymbol{d}_1'\boldsymbol{x}_1+\boldsymbol{d}_2'\boldsymbol{x}_2 = \varphi(\boldsymbol{x}_1, \boldsymbol{x}_2).$$

As in the theorem we distinguish between five cases :

(i) $\boldsymbol{D}_{22} = \boldsymbol{0}$, $\boldsymbol{D}_{12} = \boldsymbol{0}$. Then τ_s exists for all $s \geqslant 0$;

(ii) $\boldsymbol{D}_{22} = 0$, $\boldsymbol{D}_{12} \neq \boldsymbol{0}$, $\boldsymbol{d}_2 = \boldsymbol{0}$. τ_s exists $\Longleftrightarrow E(\boldsymbol{x}_1'\boldsymbol{D}_{12}\boldsymbol{x}_2/\boldsymbol{x}_1'\boldsymbol{x}_1)^s\ (\boldsymbol{d}_1'\boldsymbol{x}_1)$ exists $\Longleftrightarrow E(1/\boldsymbol{x}_1'\boldsymbol{x}_1)^{(s-1)/2}$ exists $\Longleftrightarrow s < r+1$;

(iii) $\boldsymbol{D}_{22} = \boldsymbol{0}$, $\boldsymbol{D}_{12} \neq \boldsymbol{0}$, $\boldsymbol{d}_2 \neq \boldsymbol{0}$. τ_s exists $\Leftrightarrow E(\boldsymbol{x}_1'\boldsymbol{D}_{12}\boldsymbol{x}_2/\boldsymbol{x}_1'\boldsymbol{x}_1)^s$ exists $\Leftrightarrow E(1/\boldsymbol{x}_1'\boldsymbol{x}_1)^{s/2}$ exists $\Leftrightarrow s < r$;

(iv) $\boldsymbol{D}_{22} \neq \boldsymbol{0}$, $\boldsymbol{d}_2 = \boldsymbol{0}$. τ_s exists $\Leftrightarrow E(1/\boldsymbol{x}_1'\boldsymbol{x}_1)^s\ (\boldsymbol{d}_1'\boldsymbol{x}_1)$ exists $\Leftrightarrow E(1/\boldsymbol{x}_1'\boldsymbol{x}_1)^{s-\frac{1}{2}}$ exists $\Leftrightarrow s < (r+1)/2$;

(v) $\boldsymbol{D}_{22} \neq 0$, $\boldsymbol{d}_2 \neq 0$. τ_s exists $\Leftrightarrow E(1/\boldsymbol{x}_1'\boldsymbol{x}_1)^s$ exists $\Leftrightarrow s < r/2$.

Finally in Theorem 3, we define

$$\boldsymbol{G}_{11} = \boldsymbol{P}_1'\boldsymbol{L}'\boldsymbol{C}\boldsymbol{L}\boldsymbol{P}_1,\ \boldsymbol{G}_{12} = \boldsymbol{P}_1'\boldsymbol{L}'\boldsymbol{C}\boldsymbol{L}\boldsymbol{P}_2,\ \boldsymbol{G}_{22} = \boldsymbol{P}_2'\boldsymbol{L}'\boldsymbol{C}\boldsymbol{L}\boldsymbol{P}_2,$$

so that

$$\varphi^*(\boldsymbol{y}) = \boldsymbol{y}'\boldsymbol{C}\boldsymbol{y} = \boldsymbol{x}_1'\boldsymbol{G}_{11}\boldsymbol{x}_1+2\boldsymbol{x}_1'\boldsymbol{G}_{12}\boldsymbol{x}_2+\boldsymbol{x}_2'\boldsymbol{G}_{22}\boldsymbol{x}_2 = \varphi(\boldsymbol{x}_1, \boldsymbol{x}_2).$$

Here we need to distinguish between seven cases. The details are as before and therefore omitted.

Proof of Theorem 4 : It follows from Theorem 6 of Magnus (1986) that

$$\theta_s(\boldsymbol{\mu}) = \frac{1}{(s-1)\,!} \sum_{\nu}\gamma_s(\nu) \int_0^\infty t^{s-1}\,|\,\boldsymbol{\Delta}\,|\,\psi_1(\boldsymbol{\mu})\psi_2(\boldsymbol{\mu})dt, \qquad \ldots \text{ (A.3)}$$

where

$$\psi_1(\boldsymbol{\mu}) = \exp\left(-\tfrac{1}{2}\boldsymbol{\mu}'\boldsymbol{V}\boldsymbol{\mu}\right)$$

and

$$\psi_2(\boldsymbol{\mu}) = \prod_{j=1}^{s} [\varphi_j(\boldsymbol{\mu})]^{n_j},\ \varphi_j(\boldsymbol{\mu}) = \operatorname{tr} \boldsymbol{R}^j+j\boldsymbol{\mu}'\boldsymbol{W}_j\boldsymbol{\mu}.$$

(Notice that ω and φ_j, as defined in Theorem 4, are the values of the functions ψ_2 and φ_j at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, so that $\omega = \psi_2(\boldsymbol{\mu}_0)$ and $\varphi_j = \varphi_j(\boldsymbol{\mu}_0)$.).

Since ψ_1 and ψ_2 are ∞ times continuously differentiable at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, so is θ_s. This proves (a).

Let us now prove (b) and (c). We first differentiate $\psi_1(\boldsymbol{\mu})$ twice at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ This yields

$$\left.\frac{\partial\psi_1(\boldsymbol{\mu})}{\partial\boldsymbol{\mu}}\right|_{\boldsymbol{\mu}=\boldsymbol{\mu}_0} = -\psi_1(\boldsymbol{\mu}_0)\boldsymbol{V}\boldsymbol{\mu}_0 \qquad \ldots \text{ (A.4)}$$

and

$$\left.\frac{\partial^2\psi_1(\boldsymbol{\mu})}{\partial\boldsymbol{\mu}\,\partial\boldsymbol{\mu}'}\right|_{\boldsymbol{\mu}=\boldsymbol{\mu}_0} = -\psi_1(\boldsymbol{\mu}_0)\,(\boldsymbol{V}-\boldsymbol{V}\boldsymbol{\mu}_0\boldsymbol{\mu}_0'\boldsymbol{V}). \qquad \ldots \text{ (A.5)}$$

Next we twice differentiate $[\varphi_j(\mu)]^{n_j}$, which gives

$$\left.\frac{\partial[\varphi_j(\boldsymbol{\mu})]^{n_j}}{\partial\boldsymbol{\mu}}\right|_{\boldsymbol{\mu}=\boldsymbol{\mu}_0} = 2jn_j\, r_j\, \boldsymbol{W}_j\boldsymbol{\mu}_0 \qquad \ldots \quad (A.6)$$

and

$$\left.\frac{\partial^2[\varphi_j(\boldsymbol{\mu})]^{n_j}}{\partial\boldsymbol{\mu}\partial\boldsymbol{\mu}'}\right|_{\boldsymbol{\mu}=\boldsymbol{\mu}_0} = 2jn_j\, r_j\boldsymbol{W}_j + 4j^2\, n_j(n_j-1)r_j^*\, \boldsymbol{W}_j\boldsymbol{\mu}_0\boldsymbol{\mu}_0'\boldsymbol{W}_j. \qquad \ldots \quad (A.7)$$

From (A.6) and (A.7) we obtain the first two derivatives of $\psi_2(\mu)$ at $\mu = \mu_0$:

$$\left.\frac{\partial\psi_2(\boldsymbol{\mu})}{\partial\boldsymbol{\mu}}\right|_{\boldsymbol{\mu}=\boldsymbol{\mu}_0} = 2\boldsymbol{W}\boldsymbol{\mu}_0 \qquad \ldots \quad (A.8)$$

and

$$\left.\frac{\partial^2\psi_2(\boldsymbol{\mu})}{\partial\boldsymbol{\mu}\partial\boldsymbol{\mu}'}\right|_{\boldsymbol{\mu}=\boldsymbol{\mu}_0} = 2\boldsymbol{W} + 4\sum_{j=1}^{s} j^2\, n_j(n_j-1)r_j^*\, \omega_j\boldsymbol{W}_j\boldsymbol{\mu}_0\boldsymbol{\mu}_0'\boldsymbol{W}_j$$

$$+4\sum_{\substack{i=1\\ i\neq j}}^{s}\sum_{j=1}^{s} \omega_{ij}(in_i\, r_i\boldsymbol{W}_i)\boldsymbol{\mu}_0\boldsymbol{\mu}_0'(jn_j\, r_j\, \boldsymbol{W}_j). \qquad \ldots \quad (A.9)$$

It then follows from (A.4), (A.5), (A.8) and (A.9) that

$$\left.\frac{\partial(\psi_1(\boldsymbol{\mu})\psi_2(\boldsymbol{\mu}))}{\partial\boldsymbol{\mu}}\right|_{\boldsymbol{\mu}=\boldsymbol{\mu}_0} = \psi_1(\boldsymbol{\mu}_0)\,(2\boldsymbol{W}-\omega\boldsymbol{V})\boldsymbol{\mu}_0 \qquad \ldots \quad (A.10)$$

and

$$\left.\frac{\partial^2(\psi_1(\boldsymbol{\mu})\psi_2(\boldsymbol{\mu}))}{\partial\boldsymbol{\mu}\partial\boldsymbol{\mu}'}\right|_{\boldsymbol{\mu}=\boldsymbol{\mu}_0} = \psi_1(\boldsymbol{\mu}_0)Z_s(t). \qquad \ldots \quad (A.11)$$

The first and second derivatives of $\theta_s(\boldsymbol{\mu})$ at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ now follow from (A.3), (A.10) and (A.11).

Proof of Theorem 5 : Combining Proposition 1 of Hoque, Magnus and Pesaran (1988) with (A.3) and Theorem 4 above, we obtain, provided the expectations exist, for $s = 1, 2, \ldots,$

$$E\left[\frac{\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y}}\right]^s \boldsymbol{y} = \boldsymbol{\Omega}\boldsymbol{h}_s + \theta_s\boldsymbol{\mu} \qquad \ldots \quad (A.12)$$

and

$$E\left[\frac{\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y}}\right]^s \boldsymbol{y}\boldsymbol{y}' = \theta_s(\boldsymbol{\Omega}+\boldsymbol{\mu}\boldsymbol{\mu}') + \boldsymbol{\Omega}\boldsymbol{H}_s\boldsymbol{\Omega} + \boldsymbol{\Omega}\boldsymbol{h}_s\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{h}_s'\boldsymbol{\Omega}, \ldots \quad (A.13)$$

where

$$\theta_s = \frac{\exp\left(-\frac{1}{2}\boldsymbol{\mu}^{*\prime}\boldsymbol{\mu}^*\right)}{(s-1)\,!} \sum_{\nu} \gamma_s(\nu) \int_0^\infty t^{s-1}|\boldsymbol{\Delta}|\, e^{\frac{1}{2}\boldsymbol{\xi}'\boldsymbol{\xi}}\, \omega dt \qquad \ldots \quad (A.14)$$

$$h_s = -\theta_s \Omega^{-1}\mu + L'^{-1} P Q_s \mu^* \qquad \ldots \quad (A.15)$$

$$H_s = \frac{\exp(-\tfrac{1}{2}\mu^{*\prime}\mu^*)}{(s-1)\,!} \sum_\nu \gamma_s(\nu) \int_0^\infty t^{s-1} |\Delta| e^{\frac{1}{2}\xi'\xi} Z_s \, dt \qquad \ldots \quad (A.16)$$

$$Q_s = \frac{\exp(-\tfrac{1}{2}\mu^{*\prime}\mu^*)}{(s-1)\,!} \sum_\nu \gamma_s(\nu) \int_0^\infty t^{s-1} |\Delta| e^{\frac{1}{2}\xi'\xi} (\omega\Delta^2 + 2\Delta\bar{R}_s\Delta)\, dt \qquad \ldots \quad (A.17)$$

$$\begin{aligned} Z_s = L'^{-1}P[& 2\Delta\bar{R}_s\Delta - \omega\{I - \Delta^2 - (I-\Delta^2)\mu^*\mu^{*\prime}(I-\Delta^2)\} \\ & -2(I-\Delta^2)\mu^*\mu^{*\prime}\Delta\bar{R}_s\Delta - 2\Delta\bar{R}_s\Delta\mu^*\mu^{*\prime}(I-\Delta^2) + 4\Delta Z_s^*\Delta]P'L^{-1} \end{aligned}$$

$$Z_s^* = \sum_{j=1}^{s} j^2 n_j(n_j-1) r_j^* \omega_j R^j \xi\xi' R^j + 2 \sum_{i<j} \omega_{ij} (i n_i r_i)(j n_j r_j) R^i \xi\xi' R^j$$

and

$$\bar{R}_s = \sum_{j=1}^{s} j n_j r_j \omega_j R^j .$$

To obtain (a) we now insert (A.15) in (A.12). This gives

$$E(y'Ay/y'By)^s y = LPQ_s\mu^*,$$

so that

$$E(y'Ay/y'By)^s (a'y) = a^{*\prime} Q_s \mu^* . \qquad \ldots \quad (A.18)$$

Inserting (A.17) in (A.18) and observing that

$$a^{*\prime}(\omega\Delta^2 + 2\Delta\bar{R}_s\Delta)\mu^* = f_s(t)$$

completes the proof of (a).

Similarly, to prove (b), we insert (A.14) and (A.17) in (A.15) and then (A.14), (A.15) and (A.16) in (A.13). This gives

$$E\left[\frac{y'Ay}{y'By}\right]^s yy' = \frac{\exp(-\tfrac{1}{2}\mu^{*\prime}\mu^*)}{(s-1)\,!} \sum_\nu \gamma_s(\nu) \int_0^\infty t^{s-1} |\Delta| e^{\frac{1}{2}\xi'\xi} G_s \, dt, \qquad \ldots \quad (A.19)$$

where

$$\begin{aligned} G_s &= \omega(\Omega - \mu\mu') + \Omega Z_s \Omega + LP\Delta(\omega I + 2\bar{R}_s)\Delta\mu^*\mu' \\ &\quad + \mu\mu^{*\prime}\Delta(\omega I + 2\bar{R}_s)\Delta P'L' \\ &= LP\Delta[\omega(I + \xi\xi') + 2\bar{R}_s + 2\xi\xi'\bar{R}_s + 2\bar{R}_s\xi\xi' + 4Z_s^*]\Delta P'L' . \qquad \ldots \quad (A.20) \end{aligned}$$

Postmultiplying both sides of (A.19) with C and taking the trace yields

$$E\left[\frac{y'Ay}{y'By}\right](y'Cy) = \frac{\exp(-\tfrac{1}{2}\mu^{*\prime}\mu^*)}{(s-1)\,!} \sum_\nu \gamma_s(\nu) \int_0^\infty t^{s-1} |\Delta| e^{\frac{1}{2}\xi'\xi} (\operatorname{tr} G_s C)\, dt. \qquad \ldots \quad (A.21)$$

Using (A.20) and the definitions of $\bar{R}_s$ and Z_s^*, it is easy to show that tr $G_s C = g_s(t)$. This, in conjunction with (A.21) proves (b) and concludes the proof of Theorem 5.

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