Some properties of a generalized two-error components matrix

Problem 01.5.1

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PROBLEMS AND SOLUTIONS

PROBLEMS

01.5.1. Some Properties of a Generalized Two-Error Components Matrix, proposed by Franc J.G.M. Klaassen and Jan R. Magnus. The standard two-error components model has a variance matrix where all variances (diagonal elements) are equal to each other and all covariances (off-diagonal elements) are equal. The most convenient way to write the matrix is as $\alpha J + \beta (I_T - J)$, where $J = (1/T)11'$, $1$ denotes a $T \times 1$ vector of ones, and $I_T$ denotes the identity matrix of order $T$. Consider a generalization of this matrix, namely the $T \times T$ matrix

$$\Omega = \begin{pmatrix}
\alpha_1 J_1 + \beta_1 (I_{T_1} - J_1) & \gamma_{t_1}^1 t_1^2 \\
\gamma_{t_2}^1 t_1^2 & \alpha_2 J_2 + \beta_2 (I_{T_2} - J_2)
\end{pmatrix},$$

where $t_1$ and $t_2$ denote vectors of ones of dimension $T_1$ and $T_2$, respectively, $J_1 = (1/T_1)11'$, $J_2 = (1/T_2)22'$, and $T = T_1 + T_2$. Let $\Delta = \alpha_1 \alpha_2 - \gamma^2 T_1 T_2$. Show that:

(a) The determinant of $\Omega$ is given by $|\Omega| = \beta_1^{T_1 - 1} \beta_2^{T_2 - 1} \Delta$.

(b) $\Omega$ is positive definite if and only if $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, and $\Delta$ are all positive.

(c) Assume that $\Omega$ is positive definite. Then $\Omega^p$ has the same structure as $\Omega$ for any real $p$. In particular,

$$\Omega^p = \begin{pmatrix}
\phi_1 J_1 + \beta_1^p (I_{T_1} - J_1) & \delta_{t_1}^1 t_2^2 \\
\delta_{t_2}^1 t_1^2 & \phi_2 J_2 + \beta_2^p (I_{T_2} - J_2)
\end{pmatrix}.$$

(d) Obtain $\phi_1$, $\phi_2$, and $\delta$ for the special cases $p = -1$ and $p = -\frac{1}{2}$.

(e) Consider the transformation $y = \Omega^p x$. Let $x' = (x_1', x_2')$ and $y' = (y_1', y_2')$, where $x_1$ and $y_1$ are of dimension $T_1$ and $x_2$ and $y_2$ are of dimension $T_2$. Then,

$$y_{1t} = y_1^* + \beta_1^p x_{1t} (t = 1, \ldots, T_1), \quad y_1^* = (\phi_1 - \beta_1^p) \bar{x}_1 + \delta T_2 \bar{x}_2,$$

and

$$y_{2t} = y_2^* + \beta_2^p x_{2t} (t = 1, \ldots, T_2), \quad y_2^* = (\phi_2 - \beta_2^p) \bar{x}_2 + \delta T_1 \bar{x}_1,$$

where $\bar{x}_1 = (1/T_1)11' x_1$ and $\bar{x}_2 = (1/T_2)22' x_2$. This property (for $p = -\frac{1}{2}$) is especially useful in obtaining (feasible) GLS estimates, as in Klaassen and Magnus (2001).

REFERENCES

Solution

*Econometric Theory, 18, 1274–1275 (2002)*
index $t$ is faster than the index $i$. Consider running ordinary least squares (OLS) on the original regression (3); running OLS on the within regression

$$y_{it} - \bar{y}_i = (x_{it} - \bar{x}_i)'\beta + v_{it} - \bar{v}_i, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \tag{4}$$

where $\bar{z}_i = T^{-1} \sum_{t=1}^{T} z_{it}$ for $z = y, x, v$; running OLS on the between regression

$$\bar{y}_i = \bar{x}_i'\beta + \mu_i + \bar{v}_i, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \tag{5}$$

with $T$ replications of the equation for each individual $i$; and running OLS on the generalized least squares transformed regression

$$y_{it} - \hat{\theta}\bar{y}_i = (x_{it}' - \hat{\theta}\bar{x}_i)'\beta + (1 - \hat{\theta})\mu_i + v_{it} - \hat{\theta}\bar{v}_i, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \tag{6}$$

where $\hat{\theta}$ is a consistent (as $n \to \infty$ and $T$ stays fixed) estimate of $\theta = 1 - \sigma_v/\sqrt{\sigma_e^2 + T\sigma_u^2}$. When each OLS estimate is obtained using a typical regression package, the Durbin–Watson statistic is provided among the regression output. Derive the probability limits of the four Durbin–Watson statistics, as $n \to \infty$ and $T$ stays fixed. Using the obtained result, propose an asymptotic test for individual effects based on the Durbin–Watson statistic.

**REFERENCE**


**SOLUTIONS**


(a) Denote the submatrix $\alpha_1 J_1 + \beta_1 (I_{T_1} - J_1)$ by $\Omega_{11}$ and define $\Omega_{22}$ accordingly. Assume first that $\Omega_{11}$ is nonsingular. Then, using a standard result on the determinant of a partitioned matrix (Magnus and Neudecker, 1999, p. 25),

$$|\Omega| = |\Omega_{11}| |\Omega_{22} - \gamma^2 t_2 t_1' \Omega_{11}^{-1} t_1 t_2'| = \alpha_1 \beta_1^{T_1-1} \left|\Omega_{22} - \frac{\gamma^2 T_1 T_2}{\alpha_1} J_2\right|$$

$$= \beta_1^{T_1-1} \beta_2^{T_2-1} \Delta.$$ 

Next, let $|\Omega_{11}| = 0$. Then, $\Omega_{11} + \epsilon I_{T_1} = (\alpha_1 + \epsilon) J_1 + (\beta_1 + \epsilon) (I_{T_1} - J_1)$ will be nonsingular for all $\epsilon$ sufficiently small, and the result follows by the continuity of the determinant, letting $\epsilon \to 0$. 


(b) To find the eigenvalues of $\Omega$, we notice that the matrix $\Omega - \lambda I_T$ has the same structure as $\Omega$ but with parameters $\alpha_i - \lambda$ and $\beta_i - \lambda$ instead of $\alpha_i$ and $\beta_i$ ($i = 1, 2$). Hence,

$$|\Omega - \lambda I_T| = (\beta_1 - \lambda)^{T_1-1}(\beta_2 - \lambda)^{T_2-1}((\alpha_1 - \lambda)(\alpha_2 - \lambda) - \gamma^2 T_1 T_2).$$

The eigenvalues are therefore $\beta_1$ ($T_1 - 1$ times), $\beta_2$ ($T_2 - 1$ times), and $\xi_1$ and $\xi_2$, where $\xi_1 + \xi_2 = \alpha_1 + \alpha_2$ and $\xi_1 \xi_2 = \Delta$. Because $\Omega$ is positive definite if and only if all its eigenvalues are positive, the result follows.

(c) Let $S_1$ be a $T_1 \times (T_1 - 1)$ matrix such that $S_1^T S_1 = I_{T_1-1}$ and $S_1^T t_1 = 0$. Then $S_1 S_1^T = I_{T_1} - J_1$. Let $S_2$ be defined similarly. Let $\Lambda$ denote the diagonal $T \times T$ matrix of eigenvalues $\beta_1$ ($T_1 - 1$ times), $\beta_2$ ($T_2 - 1$ times), $\xi_1$ and $\xi_2$ and define the $T \times T$ matrix

$$V = \begin{pmatrix} S_1 & 0 & \theta_1 t_1 & \omega_1 t_1 \\ 0 & S_2 & \theta_2 t_2 & \omega_2 t_2 \end{pmatrix}.$$ 

Then one verifies that $\Omega V = V \Lambda$ and $V^T V = I_T$ for suitable choices of $\theta_1$, $\omega_1$, $\theta_2$, and $\omega_2$. Hence, $\Omega^p = V \Lambda^p V^T$, and this has the same form as $\Omega$.

(d) For $p = -1$, we find

$$\phi_1 = \alpha_2 / \Delta, \quad \phi_2 = \alpha_1 / \Delta, \quad \delta = -\gamma / \Delta,$$

and for $p = -\frac{1}{2}$ we obtain

$$\phi_1 = \frac{\alpha_2 + \Delta^{1/2}}{\Delta^{1/2} \theta}, \quad \phi_2 = \frac{\alpha_1 + \Delta^{1/2}}{\Delta^{1/2} \theta}, \quad \delta = -\frac{\gamma}{\Delta^{1/2} \theta},$$

where $\theta = \sqrt{\alpha_1 + \alpha_2 + 2 \Delta^{1/2}}$.

(e) Finally, part (e) follows from (c) by direct calculation.

**REFERENCE**