Some properties of a generalized two-error components matrix

Problem 01.5.1

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PROBLEMS AND SOLUTIONS

PROBLEMS

01.5.1. *Some Properties of a Generalized Two-Error Components Matrix*, proposed by Franc J.G.M. Klaassen and Jan R. Magnus. The standard two-error components model has a variance matrix where all variances (diagonal elements) are equal to each other and all covariances (off-diagonal elements) are equal. The most convenient way to write the matrix is as $\alpha J + \beta(I_T - J)$, where $J = (1/T)\imath\imath'$, $\imath$ denotes a $T \times 1$ vector of ones, and I_T denotes the identity matrix of order T. Consider a generalization of this matrix, namely the $T \times T$ matrix

$$\Omega = \begin{pmatrix} \alpha_1 J_1 + \beta_1(I_{T_1} - J_1) & \gamma \imath_1 \imath_2' \\ \gamma \imath_2 \imath_1' & \alpha_2 J_2 + \beta_2(I_{T_2} - J_2) \end{pmatrix},$$

where $\imath_1$ and $\imath_2$ denote vectors of ones of dimension T_1 and T_2, respectively, $J_1 = (1/T_1)\imath_1\imath_1'$, $J_2 = (1/T_2)\imath_2\imath_2'$, and $T = T_1 + T_2$. Let $\Delta = \alpha_1\alpha_2 - \gamma^2 T_1 T_2$. Show that:

(a) The determinant of Ω is given by $|\Omega| = \beta_1^{T_1-1}\beta_2^{T_2-1}\Delta$.
(b) Ω is positive definite if and only if α_1, α_2, β_1, β_2, and Δ are all positive.
(c) Assume that Ω is positive definite. Then Ω^p has the same structure as Ω for any real p. In particular,

$$\Omega^p = \begin{pmatrix} \phi_1 J_1 + \beta_1^p(I_{T_1} - J_1) & \delta \imath_1 \imath_2' \\ \delta \imath_2 \imath_1' & \phi_2 J_2 + \beta_2^p(I_{T_2} - J_2) \end{pmatrix}.$$

(d) Obtain ϕ_1, ϕ_2, and δ for the special cases $p = -1$ and $p = -\frac{1}{2}$.
(e) Consider the transformation $y = \Omega^p x$. Let $x' = (x_1', x_2')$ and $y' = (y_1', y_2')$, where x_1 and y_1 are of dimension T_1 and x_2 and y_2 are of dimension T_2. Then,

$$y_{1t} = y_1^* + \beta_1^p x_{1t} (t = 1, \ldots, T_1), \qquad y_1^* = (\phi_1 - \beta_1^p)\bar{x}_1 + \delta T_2 \bar{x}_2,$$

and

$$y_{2t} = y_2^* + \beta_2^p x_{2t} (t = 1, \ldots, T_2), \qquad y_2^* = (\phi_2 - \beta_2^p)\bar{x}_2 + \delta T_1 \bar{x}_1,$$

where $\bar{x}_1 = (1/T_1)\imath_1' x_1$ and $\bar{x}_2 = (1/T_2)\imath_2' x_2$. This property (for $p = -\frac{1}{2}$) is especially useful in obtaining (feasible) GLS estimates, as in Klaassen and Magnus (2001).

REFERENCES

Klaassen, F.J.G.M. & J.R. Magnus (2001) Are points in tennis independent and identically distributed? Evidence from a dynamic binary panel data model. *Journal of the American Statistical Association* 96, forthcoming.

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Solution

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index t is faster than the index i. Consider running ordinary least squares (OLS) on the original regression (3); running OLS on the within regression

$$y_{it} - \bar{y}_{i.} = (x_{it} - \bar{x}_{i.})'\beta + v_{it} - \bar{v}_{i.}, \qquad i = 1,\ldots,n, \qquad t = 1,\ldots,T, \tag{4}$$

where $\bar{z}_{i.} = T^{-1}\sum_{t=1}^{T} z_{it}$ for $z = y, x, v$; running OLS on the between regression

$$\bar{y}_{i.} = \bar{x}_{i.}'\beta + \mu_i + \bar{v}_{i.}, \qquad i = 1,\ldots,n, \qquad t = 1,\ldots,T, \tag{5}$$

with T replications of the equation for each individual i; and running OLS on the generalized least squares transformed regression

$$y_{it} - \hat{\theta}\bar{y}_{i.} = (x_{it}' - \hat{\theta}\bar{x}_{i.})'\beta + (1-\hat{\theta})\mu_i + v_{it} - \hat{\theta}\bar{v}_{i.}, \qquad i = 1,\ldots,n, \qquad t = 1,\ldots,T, \tag{6}$$

where $\hat{\theta}$ is a consistent (as $n \to \infty$ and T stays fixed) estimate of $\theta = 1 - \sigma_v \big/ \sqrt{\sigma_v^2 + T\sigma_\mu^2}$. When each OLS estimate is obtained using a typical regression package, the Durbin–Watson statistic is provided among the regression output. Derive the probability limits of the four Durbin–Watson statistics, as $n \to \infty$ and T stays fixed. Using the obtained result, propose an asymptotic test for individual effects based on the Durbin–Watson statistic.

REFERENCE

Baltagi, B.H. (2001) *Econometric Analysis of Panel Data*. New York: Wiley.

SOLUTIONS

01.5.1. *Some Properties of a Generalized Two-Error Components Matrix*—Solution 1, proposed, by Franc J.G.M. Klaassen and Jan R. Magnus.

(a) Denote the submatrix $\alpha_1 J_1 + \beta_1(I_{T_1} - J_1)$ by Ω_{11} and define Ω_{22} accordingly. Assume first that Ω_{11} is nonsingular. Then, using a standard result on the determinant of a partitioned matrix (Magnus and Neudecker, 1999, p. 25),

$$|\Omega| = |\Omega_{11}|\,|\Omega_{22} - \gamma^2 \iota_2 \iota_1' \Omega_{11}^{-1} \iota_1 \iota_2'| = \alpha_1 \beta_1^{T_1 - 1} \left| \Omega_{22} - \frac{\gamma^2 T_1 T_2}{\alpha_1} J_2 \right|$$

$$= \beta_1^{T_1-1}\beta_2^{T_2-1}\Delta.$$

Next, let $|\Omega_{11}| = 0$. Then, $\Omega_{11} + \epsilon I_{T_1} = (\alpha_1 + \epsilon)J_1 + (\beta_1 + \epsilon)(I_{T_1} - J_1)$ will be nonsingular for all ϵ sufficiently small, and the result follows by the continuity of the determinant, letting $\epsilon \to 0$.

(b) To find the eigenvalues of Ω, we notice that the matrix $\Omega - \lambda I_T$ has the same structure as Ω but with parameters $\alpha_i - \lambda$ and $\beta_i - \lambda$ instead of α_i and β_i $(i = 1,2)$. Hence,

$$|\Omega - \lambda I_T| = (\beta_1 - \lambda)^{T_1 - 1}(\beta_2 - \lambda)^{T_2 - 1}((\alpha_1 - \lambda)(\alpha_2 - \lambda) - \gamma^2 T_1 T_2).$$

The eigenvalues are therefore β_1 ($T_1 - 1$ times), β_2 ($T_2 - 1$ times), and ξ_1 and ξ_2, where $\xi_1 + \xi_2 = \alpha_1 + \alpha_2$ and $\xi_1 \xi_2 = \Delta$. Because Ω is positive definite if and only if all its eigenvalues are positive, the result follows.

(c) Let S_1 be a $T_1 \times (T_1 - 1)$ matrix such that $S_1' S_1 = I_{T_1 - 1}$ and $S_1' \iota_1 = 0$. Then $S_1 S_1' = I_{T_1} - J_1$. Let S_2 be defined similarly. Let Λ denote the diagonal $T \times T$ matrix of eigenvalues β_1 ($T_1 - 1$ times), β_2 ($T_2 - 1$ times), ξ_1 and ξ_2 and define the $T \times T$ matrix

$$V = \begin{pmatrix} S_1 & 0 & \theta_1 \iota_1 & \omega_1 \iota_1 \\ 0 & S_2 & \theta_2 \iota_2 & \omega_2 \iota_2 \end{pmatrix}.$$

Then one verifies that $\Omega V = V\Lambda$ and $V'V = I_T$ for suitable choices of θ_1, ω_1, θ_2, and ω_2. Hence, $\Omega^p = V\Lambda^p V'$, and this has the same form as Ω.

(d) For $p = -1$, we find

$$\phi_1 = \alpha_2/\Delta, \qquad \phi_2 = \alpha_1/\Delta, \qquad \delta = -\gamma/\Delta,$$

and for $p = -\frac{1}{2}$ we obtain

$$\phi_1 = \frac{\alpha_2 + \Delta^{1/2}}{\Delta^{1/2}\theta}, \qquad \phi_2 = \frac{\alpha_1 + \Delta^{1/2}}{\Delta^{1/2}\theta}, \qquad \delta = -\frac{\gamma}{\Delta^{1/2}\theta},$$

where $\theta = \sqrt{\alpha_1 + \alpha_2 + 2\Delta^{1/2}}$.

(e) Finally, part (e) follows from (c) by direct calculation.

REFERENCE

Magnus, J.R. & H. Neudecker (1999) *Matrix Differential Calculus with Applications in Statistics and Econometrics*, rev. ed. New York: Wiley.

01.5.1 *Some Properties of a Generalized Two-Error Components Matrix*—Solution 2, proposed by Tom Wansbeek. The notation ι, J, Δ, Ω is used according to the formulation of the problem. Moreover, let $j_1 \equiv \iota_1 \big/ \sqrt{T_1}$ (so $j_1 j_1' = J_1$), G_1 is such that $G_1 \perp \iota_1, G_1' G_1 = I_{T_1}, G_1 G_1' = I_{T_1} - J_1$ (of rank $T_1 - 1$), $\gamma_* \equiv \gamma\sqrt{T_1 T_2}$ (so $\Delta = \alpha_1 \alpha_2 - \gamma_*^2$),

$$L \equiv \begin{bmatrix} j_1 & 0 & G_1 & 0 \\ 0 & j_2 & 0 & G_2 \end{bmatrix}, \qquad A \equiv \begin{bmatrix} \alpha_1 & \gamma_* \\ \gamma_* & \alpha_2 \end{bmatrix}, \qquad B \equiv \begin{bmatrix} \beta_1 I_{T_1 - 1} & 0 \\ 0 & \beta_2 I_{T_2 - 1} \end{bmatrix},$$