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Expected utility and catastrophic consumption risk☆

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HIGHLIGHTS

- We derive conditions on utility to avoid fragility of a cost-benefit analysis.
- The conditions ensure that expected (marginal) utility of consumption is finite.
- Our context-free results pertain to managing catastrophic consumption risk.

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ABSTRACT

An expected utility based cost-benefit analysis is, in general, fragile to distributional assumptions. We derive necessary and sufficient conditions on the utility function of consumption in the expected utility model to avoid this. The conditions ensure that expected (marginal) utility of consumption and the expected intertemporal marginal rate of substitution that trades off consumption and self-insurance remain finite, also under heavy-tailed distributional assumptions. Our results are relevant to various fields encountering catastrophic consumption risk in cost-benefit analysis.

1. Introduction

An economist, when asked to model decision making under risk or uncertainty for normative purposes, would typically work within the expected utility framework with constant relative risk aversion (that is, power utility). A statistician, on the other hand, would model economic catastrophes through probability distributions with heavy tails. Unfortunately, expected power utility is fragile with respect to heavy-tailed distributional assumptions: expected utility may fail to exist or it may imply 'incredible' conclusions.

Economists have long been aware of this tension between the expected utility paradigm and distributional assumptions (Menger, 1934), and the discussions in Arrow (1974), Ryan (1974),

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and Fishburn (1976) deal explicitly with the trade-off between the richness of the class of utility functions and the generality of the permitted distributional assumptions. Compelling examples in Geweke (2001) corroborate the fragility of the existence of expected power utility with respect to minor changes in distributional assumptions.

The combination of heavy-tailed distributions and the power utility family may not only imply infinite expected utility, but also infinite expected *marginal* utility, and hence, via the intertemporal marginal rate of substitution (the pricing kernel), lead to unacceptable conclusions in cost-benefit analyses. For example, with heavy-tailed log-consumption and power utility, the representative agent should postpone *any* unit of current consumption to mitigate future catastrophes. The latter aspect was recently emphasized by Weitzman (2009) in the context of catastrophic climate change. Weitzman also argues that attempts to avoid this unacceptable conclusion will necessarily be non-robust. Related questions about the validity of expected utility analysis in a catastrophic climate change context were analyzed by Chichilnisky (2000) and Tol (2003), and more recently by Horowitz and Lange (2009), Karp (2009), Arrow (2009), Nordhaus (2009, 2011), Pindyck (2011), Buchholz and Schymura (2012), and Chanel and Chichilnisky (2013), among others.

The current paper contributes to the literature on how to conduct an expected utility based cost-benefit analysis in the presence of catastrophic consumption risk, by deriving general theoretical compatibility results on the utility function of consumption in the expected utility model, leaving probability distributions unrestricted. Our theoretical results are context-free, and they are relevant to various fields encountering catastrophic consumption-risk analysis, such as risk management and self-insurance, finance, and environmental economics. More specifically, we obtain necessary and sufficient conditions on the utility function of consumption in the expected utility model which avoid the fragility of an expected utility based cost-benefit analysis to its distributional assumptions. These conditions ensure that expected utility and expected marginal utility of consumption – and hence the expected intertemporal marginal rate of substitution that trades off consumption and self-insurance – remain finite also under heavy-tailed distributional assumptions. Thus, they support a valid axiomatization of expected utility and avoid incredible consequences in a cost-benefit analysis.

We emphasize that this paper deals with the problem of intertemporal consumption choice and self-insurance in the presence of catastrophic consumption risk, and that its results cannot directly be translated to the setting of insurance premium calculation (Goovaerts et al., 1984; Kaas et al., 2008, Chapter 1). We expand on this later in the paper.

The remainder of the paper is organized as follows. Section 2 lists four principles on which the paper is built. Section 3 introduces the basic setting and notation. Section 4 presents the optimal consumption and self-insurance model used to conduct cost-benefit analysis. Section 5 studies expected (marginal) utility and catastrophic consumption risk within this model, deriving results on the trade-off between permitted distributional assumptions and the existence of expected (marginal) utility of consumption. Section 6 provides some examples. Section 7 generalizes the main result of Section 5 to arbitrary order of differentiation. Section 8 concludes. Proofs are relegated to Appendix.

2. Four principles

Our paper is built on four principles, which will recur in our analysis:

(i) *Catastrophic risks are important*. To study risks that can lead to catastrophe is important in many areas, e.g., financial (insurer, pension, bank, trader) distress, traffic accidents (bridge collapse, airplane crash, flight control system failure), dike bursts, killer asteroids, nuclear power plant disasters, and extreme climate change. Such low-probability high-impact events should not be ignored in cost-benefit analyses for policy making.

(ii) *Light-tailed risk may lead to heavy-tailed risk*. When x is normally distributed (light tails), then e^x has finite moments. But when x follows a Student distribution (heavy tails), then e^x has no finite moments. Light-tailed risk can easily generate heavy-tailed risk. For example, in classical Bayesian statistics, the posterior predictive distribution of an initially light-tailed distribution, often has heavy tails. A prototypical example is the case of a normal distribution, which is light-tailed if its standard deviation is known, but becomes heavy-tailed Student if uncertainty about its standard deviation is integrated out against a standard scale-invariant non-informative prior (see, e.g., Geweke, 2001; Weitzman, 2009). Even if initial processes do not have heavy tails in their distribution, this does not guarantee that consequences of these processes cannot have heavy tails. Therefore, in the presence of uncertainty, it may well be reasonable to use heavy-tailed distributional assumptions to model future (log)-consumption.

(iii) *The price to reduce catastrophic risk is finite*. Are we willing to spend all our wealth to avoid children being killed at a dangerous street corner? Or dikes to burst? Or a power plant to explode? Or a killer asteroid to hit the Earth? Or climate to change rapidly? No, we are not. To assume the opposite (that a society would be willing to offer all of its current wealth to avoid, mitigate, or self-insure against catastrophic risks) is not credible, not even from a normative perspective. There is a limit to the amount of current consumption that the representative agent is willing to give up in order to obtain one additional *certain* unit of future consumption, no matter how extreme and irreversible a catastrophic risk may be. In other words: the expected pricing kernel is finite.

(iv) *A good model 'in the center' is not necessarily good 'at the edges'*. Suppose we have estimated a function $C = a + bY$, relating consumption C to disposable income Y. The dots in Fig. 1 represent the data and the line gives the resulting least-squares prediction $\hat{C} = \hat{a} + \hat{b}Y$. For incomes in the center, roughly between 40 and 80, the consumption function can be well approximated by the regression line. How useful is this result for very low (or very high) incomes? Not very useful. For very low incomes, predicted consumption would even become negative! This does not mean that a linear consumption function is useless, but it is only useful in the center of the domain. Models are approximations, not truths (cf. Goovaerts et al., 2010, p. 301), and approximations may not work well if we move too far away from the point of approximation. Examples are abundant and easy to find. Newton's theory works fine for cars and trains, but not for space ships. Pharmaceutical testing is typically performed on adult men, and may (and often does) work differently for women and children (Litt, 1997). In quantitative risk management, it has become common practice to use separate models for the central part of the data and for the extremes, and to 'glue' the models together at a carefully chosen order statistic; see Peng (2001), Johansson (2003), and Necir and Meraghni (2009), and the references therein.

In what follows we accommodate principles (i) and (ii) by leaving distributional assumptions unrestricted. We account for (iii) by ensuring that expected (marginal) utility of consumption remains finite. Our necessary and sufficient conditions (to be presented shortly) imply that the widely adopted power utility function should not be used with catastrophic (heavy-tailed) consumption

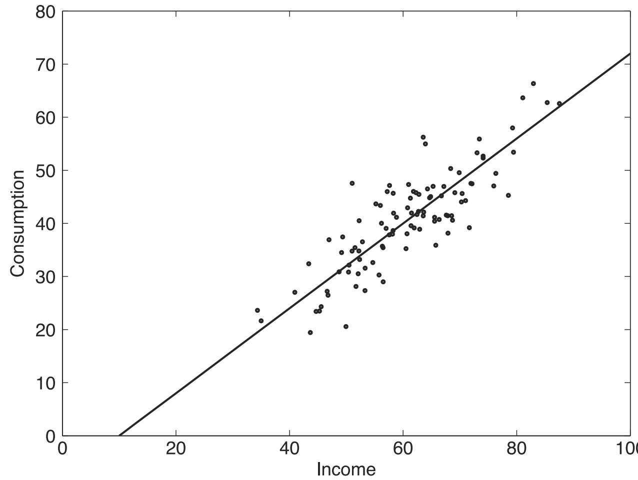

Fig. 1. A consumption function.

risks that have non-negligible support 'at the edges', thus confirming (iv). Instead, exponential utility (see, e.g., Gerber, 1979, Chapter 5, Goovaerts et al., 2004; Laeven and Stadje, 2013) or Pareto utility (Ikefuji et al., 2013) are more appropriate.

3. Setting and notation

We formulate our cost-benefit analysis as a decision under uncertainty problem, in Savage (1954) style. We fix a set $\mathcal{S}$ of states of nature and we let $\mathcal{A}$ denote a σ-algebra of subsets of $\mathcal{S}$. One state is the true state. We also fix a set $\mathcal{C}$ of consequences (outcomes, i.e. consumption) endowed with a σ-algebra $\mathcal{F}$. Since we are only interested in monetary outcomes, we may take $\mathcal{C} = \mathbb{R}_+$. A decision alternative (policy bundle) X is a measurable mapping from $\mathcal{S}$ to $\mathcal{C}$, so that $X^{-1}(E) \in \mathcal{A}$ for all events $E \in \mathcal{F}$. We assume that the class of all decision alternatives $\mathcal{X}$ is endowed with a preference order $\succeq$.

Definition 3.1. We say that expected utility (EU) holds if there exist a measurable and strictly increasing function $U : \mathcal{C} \to \mathbb{R}$ on the space of consequences, referred to as the utility function, and a probability measure $\mathbb{P}$ on $\mathcal{A}$, such that the preference order $\succeq$ on $\mathcal{X}$ is represented by a functional V of the form $X \mapsto \int_{\mathcal{S}} U(X(s))\, d\mathbb{P} = EU(X) = V(X)$. Thus, the decision alternative $X \in \mathcal{X}$ is preferred to the decision alternative $Y \in \mathcal{X}$ if, and only if, $V(X) \geq V(Y)$.

In the Von Neumann and Morgenstern (1944) framework, the utility function U is subjective, whereas the probability measure $\mathbb{P}$ associated with $\mathcal{A}$ is objective and known beforehand (decision under risk). In the more general framework of Savage (1954) adopted here, the probability measure itself can be, but need not be, subjective (decision under uncertainty). We henceforth assume that U is defined for $x \geq 0$, twice differentiable, and such that $U'(x) > 0$ and $U''(x) < 0$ for $x > 0$. We will take the perspective of an individual representative agent. The agent is concerned with his (or her) consumption choice and asks how much current consumption he is willing to give up to mitigate future catastrophic consumption losses, as formalized via his intertemporal marginal rate of substitution (see Eq. (3)). Consumption is restricted to be non-negative, so that U is defined on the positive real line. Note that in the (different) context of the problem of insurance premium calculation (Goovaerts et al., 1984; Kaas et al., 2008, Chapter 1), we would need the utility function U to be defined also on the negative real line. We do not consider this setting in the current paper.

Since the axiomatization of EU by Von Neumann and Morgenstern (1944) and Savage (1954), a variety of objections have been raised against it. These objections relate primarily to the empirical evidence that the behavior of agents under risk and uncertainty does not agree with EU. Despite important developments in non-expected utility theory, EU remains the dominant *normative* decision theory (Broome, 1991; Sims, 2001), and the current paper stays within the framework of EU. The results presented below provide compatibility conditions under which expected utility theory may reliably provide normatively appealing results, also in the presence of catastrophic risks. Of course, one may legitimately question whether EU is the appropriate normative theory for decision making under catastrophic risks and (continue a) search for better theories; see, e.g., Chichilnisky (2000). This, however, is beyond our current scope.

Definition 3.2. We say that a risk $\epsilon : \mathcal{S} \to \mathbb{R}$ is heavy-tailed to the left under $\mathbb{P}$ if its moment-generating function is infinite: $\mathrm{E}\,(e^{\gamma\epsilon}) = \infty$ for any $\gamma < 0$.

Examples of heavy-tailed risks are the Student, (mirrored) log-normal, and (mirrored) Pareto distributions. Heavy-tailed risks provide appropriate mathematical models for low-probability high-impact events, such as actuarial, financial, or environmental catastrophes; see, e.g., Laeven et al. (2005) and the references therein.

4. Optimal consumption and self-insurance under catastrophic consumption risk

Consider a simple two-period setting running from time $t = 0$ (the present) to time $t = 1$ (the distant future). By distinguishing between the present and the future, we offer agents the possibility to adjust their level of consumption in the presence of future potentially catastrophic consumption risks. This is achieved by consuming less now, and postpone consumption in order to self-insure against future consumption risks.

In this two-period setting we consider a standard representative agent with *time-additive* EU preferences, a subjective utility function U_t $(t = 0, 1)$, an objectively or subjectively given probability measure $\mathbb{P}$, and time-preference parameter $\rho > 0$. In the time-additive EU setting, U_t is sometimes referred to as the felicity (or momentary utility or subutility) function; see, e.g., Gollier (2001, Part VI, Section 15.2, p. 218). We write $U_0(x) = U(x)$ and assume that $U_1(x) = U(x)/(1 + \rho)$.

At time $t = 0$, our representative agent maximizes expected utility from certain consumption C_0 at the present time and uncertain consumption C_1 at the future time:

$$V(C_0, C_1) = U(C_0) + \frac{1}{1+\rho}\mathrm{E}U\,(C_1)\,. \qquad (1)$$

The consumption choice (C_0, C_1) is restricted to a budget-feasible consumption set. We suppose that the agent is endowed with initial wealth $W_0 > 0$, that $0 \leq C_0 \leq W_0$, and that there is no additional income or endowment at time $t = 1$. Furthermore, we suppose that all wealth not consumed at the present time $t = 0$ is transferred to the future time $t = 1$, at which the agent faces consumption risk. Specifically, we suppose that the uncertain future wealth not consumed today, W_1, is given by

$$W_1 = (W_0 - C_0)\exp(A\epsilon_1), \quad A > 0,$$

for a risk ϵ_1 with a (subjectively or objectively) given probability distribution under $\mathbb{P}$. The future wealth of the agent depends on the realization of the economy's state of nature, with an associated gross return of $\exp(A\epsilon_1)$ at time $t = 1$, together with the consumption choice made at time $t = 0$. Then, C_1 is restricted to a budget-feasible consumption set that is subject to uncertainty (ϵ_1):

$$0 \leq C_1(\epsilon_1) \leq W_1 = (W_0 - C_0)\exp(A\epsilon_1).$$

Because U is defined only on the positive real line, the non-negativity constraint imposed on uncertain future consumption C_1 is a necessary assumption for the main results in this paper.

Thus, we see that the budget restriction on C_1 takes the generic form:

$$0 \leq C_1(\epsilon_1) \leq B\exp(A\epsilon_1), \quad B \geq 0. \tag{2}$$

We exploit (2) to derive compatibility conditions on the utility function of consumption. (Henceforth, we often omit the dependence of C_1 on ϵ_1 for notational convenience.)

We now state the following definition, formalizing the term catastrophic consumption risk:

Definition 4.1. We say that the representative agent faces *catastrophic consumption risk* at the future time $t = 1$ when ϵ_1 is heavy-tailed to the left under the (objectively or subjectively) given probability measure $\mathbb{P}$.

To derive the optimal consumption bundle (C_0^*, C_1^*) in (1), we consider the first-order condition for individual optimality, given by

$$U'(C_0^*) = \frac{1}{1+\rho}\mathrm{E}\left(U'((W_0 - C_0^*)\exp(A\epsilon_1))\exp(A\epsilon_1)\right).$$

In view of the fact that in optimality $C_1^* = (W_0 - C_0^*)\exp(A\epsilon_1)$, the first-order condition can be rewritten as

$$1 = \frac{1}{1+\rho}\mathrm{E}\left(\frac{U'(C_1^*)\exp(A\epsilon_1)}{U'(C_0^*)}\right)$$
$$= \mathrm{E}\left(P_{C_0^*}(C_1^*)\exp(A\epsilon_1)\right),$$

with intertemporal marginal rate of substitution (or pricing kernel)

$$P_{C_0^*}(C_1^*) = \frac{U'(C_1^*)}{(1+\rho)U'(C_0^*)}.$$

Thus, the intertemporal marginal rate of substitution, $P_{C_0^*}(C_1^*)$, appears as part of the first-order condition for individual optimality in a standard problem of intertemporal consumption choice under uncertainty. More generally, the expectation $\mathrm{E}(P_{C_0^*}(C_1^*)X_1)$ can be regarded as the price at the present time $t = 0$ of a contingent payment X_1 at time $t = 1$.

Henceforth, we normalize without loss of generality the agent's time $t = 0$ consumption, C_0, by setting $C_0 = 1$, and we define the associated pricing kernel (or intertemporal marginal rate of substitution) as follows:

$$P(C_1^*) = \frac{U'(C_1^*)}{(1+\rho)U'(1)}. \tag{3}$$

(We will sometimes suppress the dependence of P on C_1^* for notational convenience.) The expectation $\mathrm{E}(P)$ represents the amount of consumption at time $t = 0$ that the representative agent is willing to give up in order to obtain one additional certain unit of consumption at time $t = 1$. The analysis that follows is based on this standard indicator. Its expectation measures, as a 'shadow price', the amount of current consumption the individual is willing to trade off against one additional certain unit of consumption at the future time when potentially catastrophic consumption risk resolves (see also Weitzman, 2009).

The intertemporal dimension of consumption risk that we consider is economically and practically relevant, and absent in the (static) EU setting that is typically considered in standard insurance premium calculation (see, e.g., Kaas et al., 2008, Chapter 1). The option to postpone current consumption is a form of self-insurance against future potentially catastrophic consumption risk: consume less at present and postpone consumption in the presence of high levels of future consumption risk.

5. Main results

Before stating our main results, we first state the following result, which dates back to Menger (1934).

Proposition 5.1. *If EU is to discriminate unambiguously among all possible alternative outcome distributions, the utility function must be bounded.*

Proposition 5.1 tells us that the EU functional is finite, hence unambiguous, for all outcome distributions if, and only if, the utility function is bounded. Moreover, it follows directly that the axiomatization of EU is valid for all outcome distributions only if the utility function is bounded. The implications are non-trivial: boundedness of the utility function must hold not just in exotic situations but also in more familiar and economically relevant settings involving high levels of uncertainty. Only a combination of utility function and outcome distribution that leads to finite expected utility is covered by the axiomatic justification of EU.

The result of Proposition 5.1 will not be required in the propositions on the finiteness of expected *marginal* utility of consumption that follow. Specifically, boundedness of the utility function of consumption will not be part of the hypotheses of our main results in Propositions 5.2, 5.3 and 7.1.

Let $\mathrm{RRA}(x) = -xU''(x)/U'(x)$ and $\mathrm{ARA}(x) = -U''(x)/U'(x)$ denote relative and absolute risk aversion, respectively, and let

$$\alpha^* = \inf_{x>0}\mathrm{RRA}(x), \qquad \beta^* = \sup_{x>0}\mathrm{ARA}(x).$$

The following result states that the expectation of the pricing kernel is finite for all outcome distributions whenever the concavity index $\mathrm{ARA}(x)$ is bounded.

Proposition 5.2. *Consider the setting of Section 4.*

(a) *If $\alpha^* > 0$ and ϵ_1 is heavy-tailed to the left under $\mathbb{P}$, then $\mathrm{E}(P) = \infty$;*
(b) *If $\beta^* < \infty$ and $\alpha^* = 0$, then $\mathrm{E}(P) < \infty$ for any ϵ_1.*

If the EU maximizer has decreasing absolute risk aversion and increasing relative risk aversion, as is commonly assumed (Eeckhoudt and Gollier, 1995, Section 4.2, Hypotheses 4.1 and 4.2), a complete and elegant characterization of boundedness of the expected pricing kernel can be obtained, as follows.

Proposition 5.3. *Consider the setting of Section 4. Assume that $\mathrm{RRA}(x)$ exists and is non-negative and non-decreasing for all $x \geq 0$ and that $\mathrm{ARA}(x)$ is non-increasing for all $x > 0$. Then, $\mathrm{E}(P) < \infty$ for any ϵ_1 if and only if $\int_0^\gamma \mathrm{ARA}(x)\,dx < \infty$ for some $\gamma > 0$.*

The above propositions provide necessary and sufficient conditions on the utility function of consumption to ensure that expected utility and expected marginal utility of consumption (hence the expected pricing kernel) are finite, also in the presence of heavy tails. These compatibility results are readily applicable to standard problems of intertemporal consumption choice and self-insurance under uncertainty. The importance of the results lies in the fact that (a) if (minus) expected utility is infinite, the axiomatic justification of EU is not valid, and (b) if the expected pricing kernel is infinite, then the amount of consumption at time $t = 0$ which the representative agent is willing to give up in order to obtain one additional certain unit of consumption at time $t = 1$ is infinite, which is not credible in most settings.

Weitzman (2009) recently noted, in a highly stylized setting of extreme climate change, that heavy-tailed uncertainty and power utility are incompatible, since this combination of uncertainty and preferences implies an infinite expected pricing kernel. To avoid this, Weitzman introduces a lower bound on consumption, argues that this lower bound is related to a parameter that resembles the value of a statistical life (VSL), and proves that the expected pricing kernel approaches infinity as the value of this parameter approaches infinity (the 'dismal theorem'). Weitzman further argues that this VSL-like parameter is hard to know.

Incompatible pairs of utility functions and distribution functions indeed exist, in the sense that the expected pricing kernel or other important policy variables become infinite. In fact, our results above provide necessary and sufficient conditions on the utility function of consumption for the expected pricing kernel to exist, also under heavy tails. But we object to the dismal theorem for the following reason: our results explicate that the dismal theorem is based on an ex ante incompatible model specification. It is avoided when the economic model (utility function) is compatible with the statistical model (heavy tails). Only then Savage's axiomatization of EU is valid.

6. Discussion and examples

Proposition 5.3 emphasizes the importance of the integral

$$I(\gamma) = \int_0^\gamma \mathrm{ARA}(x)\, dx.$$

If $I(\gamma) = \infty$ for some $\gamma > 0$, then $\beta^* = \infty$, but both $\alpha^* > 0$ and $\alpha^* = 0$ can occur. If $I(\gamma) = \infty$ and $\alpha^* > 0$, then we do not need the full force of Proposition 5.3; it is sufficient that ϵ_1 is heavy-tailed to the left. Then $\mathrm{E}(P) = \infty$ by Proposition 5.2(a). An example is provided by the *power* utility function

$$U(x) = \frac{x^{1-\alpha} - 1}{1-\alpha} \quad (\alpha > 0),$$

where RRA $= \alpha$ (constant), so that $\alpha^* = \alpha > 0$.

If $I(\gamma) = \infty$ and $\alpha^* = 0$, then heavy-tailedness alone is not sufficient, but we can always find an ϵ_1 such that $\mathrm{E}(P) = \infty$. An example of this situation is a utility function which satisfies the conditions of the proposition and where the ARA behaves like $-1/(x \log x)$ for values of x close to 0.

Next consider the case where $I(\gamma) < \infty$ for some $\gamma > 0$. Then $\alpha^* = 0$, but both $\beta^* < \infty$ and $\beta^* = \infty$ can occur. An example is provided by the *gexpo* utility function (Ikefuji et al., 2013, Section 5),

$$U(x) = 1 - \frac{x h^{-p}(x)\, \Gamma(p, ph(x))}{\lambda \Gamma(p)} \quad (p \geq 1, \lambda > 0),$$

where $h(x) = (x/\lambda)^{1/p}$, $\Gamma(p,h) = \int_h^\infty t^{p-1} e^{-t}\, dt$ denotes the incomplete gamma function, and $\Gamma(p) = \Gamma(p, 0)$ is the (complete) gamma function.

With gexpo utility we find $\mathrm{ARA}(x) = h(x)/x$, so that $I(\gamma)$ remains finite for any $\gamma > 0$. When $p > 1$ we have $\beta^* = \infty$, but when $p = 1$ gexpo utility specializes to *exponential* utility,

$$U(x) = 1 - e^{x/\lambda} \quad (\lambda > 0),$$

for which $\mathrm{ARA}(x) = 1/\lambda$ (constant), so that $\beta^* = 1/\lambda < \infty$.

The gexpo utility function satisfies the three properties

- P1: ARA is non-increasing and convex, and RRA is non-decreasing and concave;
- P2: Expected utility is finite irrespective of the probability distribution; and
- P3: Expected marginal utility is finite irrespective of the probability distribution;

but it does not satisfy

- P4: Utility behaves power-like for inputs remote from 0: $\mathrm{RRA}(\infty) < \infty$.

Property P1 is in the spirit of Arrow; see also Eeckhoudt and Gollier (1995, Section 4.2), P2 is satisfied if and only if utility is bounded from above and below, P3 is satisfied if and only if $I(\gamma) < \infty$ for some $\gamma > 0$, and P4 is satisfied if and only if $\mathrm{RRA}(x)$ remains finite as $x \to \infty$.

A utility function satisfying all *four* properties is the *Pareto* utility function (Ikefuji et al., 2013),

$$U(x) = 1 - (1 + x/\lambda)^{-k} \quad (k > 0,\ \lambda > 0).$$

In the Pareto family of utility functions both $I(\gamma)$ and β^* are finite. The function was introduced in order to avoid the drawbacks of both power and exponential utility 'near the edges'. Power utility is often appropriate for 'large' inputs x, but exponential utility less so because its RRA increases without bound. Exponential utility, on the other hand, is often appropriate in near-catastrophe cases ('small' x), but power utility less so because of its extreme behavior near $x = 0$. Pareto utility is a flexible yet simple and parsimonious family of utility functions, which behaves credibly for *all* non-negative inputs. Due to Properties P2 and P3, it is particularly suited for heavy-tailed risk analysis.

7. Generalization to arbitrary order

Assume U is differentiable for all degrees of differentiation. We denote by $U^{(m)}$ the mth derivative of U, $m = 0, 1, \ldots$. Furthermore, assume $(-1)^m U^{(m)}(x) < 0$, for $x > 0$ and $m = 0, 1, \ldots$. Proposition 5.3 readily generalizes to the following result, valid for any order of differentiation:

Proposition 7.1. *Arbitrarily fix $m = 0, 1, \ldots$. Assume that time-additive EU holds and that the budget feasibility restriction (2) applies. Assume furthermore that $-xU^{(m+1)}(x)/U^{(m)}(x)$ exists and is non-negative and non-decreasing for all $x \geq 0$ and that $-U^{(m+1)}(x)/U^{(m)}(x)$ is non-increasing for all $x > 0$. Then,*

$$\mathrm{E}\left(U^{(m)}(C_1^*)/U^{(m)}(1)\right) < \infty$$

for any ϵ_1 if and only if

$$\int_0^\gamma -U^{(m+1)}(x)/U^{(m)}(x)\, dx < \infty$$

for some $\gamma > 0$.

We note several special cases. For $m = 0$, $-U^{(1)}(x)/U^{(0)}(x)$ is connected to the (reciprocal of the) fear of ruin index (Foncel and Treich, 2005); for $m = 1$, $-U^{(2)}(x)/U^{(1)}(x)$ is the $\mathrm{ARA}(x)$; and for $m = 2$, $-U^{(3)}(x)/U^{(2)}(x)$ is the index of absolute prudence (see, e.g., Gollier, 2001, p. 265).

8. Concluding remarks

We have derived necessary and sufficient conditions on the utility function of consumption to avoid fragility to distributional assumptions in an EU based cost-benefit analysis. Our generic results regarding the relationship between the richness of the class of utility functions and the generality of the permitted distributional assumptions imply a simple rule for applied research, namely to only allow utility functions of consumption that are compatible with distributional assumptions. This rule supports a valid axiomatization of EU and avoids the unacceptable conclusion that society should sacrifice an unlimited amount of consumption to reduce the probability of an economic catastrophe by even a small amount.

The analysis in our paper applies to various policy making settings involving catastrophic consumption risks, such as the design of individual financial pension plans that smooth the impact of large financial shocks on consumption over time, cost-benefit analyses concerning medical risks (pandemic flu and vaccination risks), and economy-climate catastrophe.

In future work, it would be of interest to study the same problem for the dual theory of choice under risk (Yaari, 1987), instead of for EU. Also, one could assume more structure on the permitted stochasticity (yet still allow for heavy tails), such as the existence of some moments as in Arrow (1974), in order to broaden the constraints on the utility function of consumption. Finally, one could investigate the analogous (but very different) compatibility problem in the context of insurance premium calculation and risk measurement using the equivalent (or zero) utility principle or the certainty equivalent (or mean value) principle see Goovaerts et al., 1984; Denuit et al., 2006), which would require the utility function to be defined also on the negative real line.

Appendix. Proofs

Proof of Proposition 5.1. See Menger (1934, p. 468) in the context of St. Petersburg-type lotteries, and also Arrow (1974, p. 136) and Gilboa (2009, pp. 108–109). Menger (implicitly) assumes boundedness from below and demonstrates that boundedness from above should hold, and it is straightforward to generalize his result to an a priori unrestricted setting.

Proof of Proposition 5.2. Let $\alpha^* > 0$. The EU maximizer is then more risk averse in the sense of Arrow–Pratt than an agent with power (CRRA) utility of index α^*. It follows from (3) that

$$\frac{P'(C_1^*)}{P(C_1^*)} = \frac{U''(C_1^*)}{U'(C_1^*)} = -\text{ARA}(C_1^*).$$

Since $\text{ARA}(x) = \text{RRA}(x)/x \geq \alpha^*/x$, we then have

$$\begin{aligned}
\text{E}(P) &= \frac{1}{1+\rho}\text{E}\exp\left(-\int_{C_1^*}^1 d\log P(x)\right) \\
&= \frac{1}{1+\rho}\text{E}\exp\left(\int_{C_1^*}^1 \text{ARA}(x)\,dx\right) \\
&\geq \frac{1}{1+\rho}\int_{C_1^*\leq 1}\exp\left(\int_{C_1^*}^1 (\alpha^*/x)\,dx\right)dF(\epsilon_1) \\
&= \frac{1}{1+\rho}\int_{C_1^*\leq 1}(C_1^*)^{-\alpha^*}\,dF(\epsilon_1) \\
&\geq \frac{B^{-\alpha^*}}{1+\rho}\int_{C_1^*\leq 1} e^{-A\alpha^*\epsilon_1}\,dF(\epsilon_1) = \infty,
\end{aligned}$$

using (2) and the fact that ϵ_1 is heavy-tailed to the left. This proves part (a). Intuitively, if agent 1 is more risk averse in the sense of Arrow–Pratt than agent 2, and if it is optimal to postpone all consumption for agent 2, then this will also be optimal for agent 1.

Next let $\alpha^* = 0$ and $\beta^* < \infty$. The EU maximizer is then less risk averse in the sense of Arrow–Pratt than an agent with exponential (CARA) utility of index β^*. Since $\alpha^* = 0$, we have $0 \leq \text{ARA}(x) \leq \beta^*$ and hence

$$\begin{aligned}
\text{E}(P) &= \int_{C_1^*\leq 1} P\,dF(\epsilon_1) + \int_{C_1^*>1} P\,dF(\epsilon_1) \\
&\leq \frac{1}{1+\rho}\int_{C_1^*\leq 1}\exp\left(\int_{C_1^*}^1 \beta^*\,dx\right)dF(\epsilon_1) \\
&\quad + \frac{1}{1+\rho}\int_{C_1^*>1}\exp\left(-\int_1^{C_1^*}\text{ARA}(x)\,dx\right)dF(\epsilon_1) \\
&\leq \frac{e^{\beta^*}\mathbb{P}(C_1^*\leq 1) + \mathbb{P}(C_1^*>1)}{1+\rho} < \infty.
\end{aligned}$$

Proof of Proposition 5.3. To prove the 'only if' part, we assume that $\int_0^\gamma \text{ARA}(x)\,dx$ is infinite for every $\gamma > 0$, and then show that there exist $(\mathcal{S}, \mathcal{A}, \mathbb{P})$ and ϵ_1 defined on it such that $\text{E}(P) = \infty$. We note that $\beta^* = \infty$. Define a function $g : (0, 1] \to [1, \infty)$ by

$$g(y) = \exp\left(\int_y^1 \text{ARA}(x)\,dx\right).$$

Then,

$$\text{E}(P) \geq \frac{1}{1+\rho}\int_{C_1^*\leq 1} g(\min(C_1^*, 1))\,dF(\epsilon_1).$$

Recall from (2) that $C_1^* \leq Be^{A\epsilon_1}$, and let ϵ_1^* be such that $Be^{A\epsilon_1^*} = 1$, so that $0 < Be^{A\epsilon_1} \leq 1$ if and only if $\epsilon_1 \leq \epsilon_1^*$. Define $u : (-\infty, \infty) \to [0, \infty)$ by

$$u(\epsilon_1) = \begin{cases} g(Be^{A\epsilon_1}) - 1 & \text{if } \epsilon_1 \leq \epsilon_1^*, \\ 0 & \text{if } \epsilon_1 > \epsilon_1^*. \end{cases}$$

Since $\text{ARA}(1) > 0$, g is monotonically decreasing and we obtain

$$\begin{aligned}
&\int_{C_1^*\leq 1} g(\min(C_1^*, 1))\,dF(\epsilon_1) \geq \int_{\epsilon_1\leq\epsilon_1^*} g(Be^{A\epsilon_1})\,dF(\epsilon_1) \\
&= \int_{\epsilon_1\leq\epsilon_1^*}(u+1)\,dF(\epsilon_1) = \text{E}(u) + \mathbb{P}(\epsilon_1\leq\epsilon_1^*).
\end{aligned}$$

Strict monotonicity of g implies its invertibility. Hence we can choose u to be any non-negative random variable whose expectation does not exist (for example, the absolute value of a Cauchy distribution), and then define ϵ_1 through $B_1 e^{\tau\epsilon_1} = g^{-1}(u+1)$. With such a choice of ϵ_1 we have $\text{E}(P) = \infty$.

To prove the 'if'-part we assume that $\int_0^\gamma \text{ARA}(x)\,dx$ is finite. This implies that $\int_0^1 \text{ARA}(x)\,dx$ is finite, so that

$$\begin{aligned}
\text{E}(P) &= \frac{1}{1+\rho}\int_{C_1^*\leq 1}\exp\left(\int_{C_1^*}^1\text{ARA}(x)\,dx\right)dF(\epsilon_1) \\
&\quad + \frac{1}{1+\rho}\int_{C_1^*>1}\exp\left(-\int_1^{C_1^*}\text{ARA}(x)\,dx\right)dF(\epsilon_1) \\
&\leq \frac{\mathbb{P}(C_1^*\leq 1)}{1+\rho}\exp\left(\int_0^1\text{ARA}(x)\,dx\right) + \frac{\mathbb{P}(C_1^*>1)}{1+\rho} < \infty,
\end{aligned}$$

using the fact that $\alpha^* = \text{RRA}(0) = 0$.

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