Econometric Theory, **22**, 2006, 968–972. Printed in the United States of America.
DOI: 10.1017/S0266466606000454

MATRIX ALGEBRA

by Karim M. Abadir and Jan R. Magnus
Cambridge University Press, 2005

REVIEWED BY
TAISUKE OTSU
Yale University

1. OVERVIEW

This book, *Matrix Algebra*, is the first volume of the Econometric Exercises series. Other volumes of the series (*Statistics* [vol. 2], *Econometric Theory I* [vol. 3], and *Empirical Applications* [vol. 4]) are to appear in the near future. A variety of planned titles will also follow (e.g., *Time Series*, *Microeconometrics*, and *Financial Econometrics*). Although there have been a few worked-exercise books in econometrics (e.g., Phillips and Wickens, 1978), this series is the first comprehensive exercise series in modern econometrics that covers the materials for advanced undergraduate and graduate econometrics courses. According to the preface, each volume follows the same format. Chapters start with a brief overview, and then exercises and solutions follow. Solutions are provided immediately after the exercise is posed. There is a variety of exercises from proofs and applications of theorems to numerical examples. In this review I discuss only the first volume in the Econometric Exercises series and find it very useful and successful. I expect succeeding volumes in the series to be of similar quality and usefulness, given the overall objectives of the series.

This book covers topics in matrix algebra that appear in undergraduate and graduate econometrics courses. The coverage is comprehensive, and the topics are thoughtfully organized from econometricians' perspectives. This book can serve as a collection of supplementary exercises for coursework or self-study. Instructors can pick and modify questions from the list of exercises for problem sets or exams. Self-learners can enjoy the advantage of this self-contained book whose exercises are accompanied by detailed solutions. In addition, I found three outstanding features that make this book remarkably novel and ambitious. First, compared to standard theorem-proof-style textbooks (e.g., Harville, 1997; Horn and Johnson, 1990), this book reorganizes the materials into the exercise-solution style and carries them out in a unique way. Whereas theorem-proof-style textbooks just let us read and follow the theorems and proofs, this textbook requires us to actually prove the theorems. The latter process is, of course, more effective for learning, though it is more challenging and time consuming. The authors successfully facilitate this learning-by-doing

Address correspondence to Taisuke Otsu, Cowles Foundation, Yale University, P.O. Box 208281, New Haven, CT 06520-8281; e-mail: taisuke.otsu@yale.edu.

  0266-4666/06 $12.00

process by laying out the exercises in increasing order of difficulty. Proofs of theorems are split into small key step questions, and most of the theorems are accompanied by their applications and numerical examples. In this way, students can learn the materials step by step at a steady and comfortable pace. Second, compared to handbooks that exhibit a collection of theoretical results (e.g., Lütkepohl, 1996), this book presents a collection of *methods* that lead to those results. When we write a research paper, it is seldom that existing results are directly applicable. We almost always have to modify or combine the existing results to create a result we need. This exercise-solution-style textbook is suitable for training in the techniques of creating new results. In that sense, this book can also be useful for doctoral students who are going to write a research paper. Third, this book is ideal for an instructor's manual for matrix algebra. Each section begins with a simple example. Next it lets us work on a special case of a theorem. Then, the theorem is shown step by step, and applications of the theorem follow. We can see exactly the same process in our classrooms. Therefore, not only the exercises but also the composition of the book are helpful for teaching. Another interesting feature is that each chapter ends with Notes that contain footnotes and historical remarks. I find that these historical remarks are useful for a short and refreshing digression in class.

The only concern I must mention is that, except for Chapter 13 (matrix calculus), this book does not have enough econometric or statistical examples. Even though all materials in this book are apparently essential for learning econometrics, students are more likely to be motivated if the book provides, at least, a short note on the potential areas of application for each topic. However, I believe this issue is not a permanent one and will be solved in the following volumes.

2. REVIEW OF CHAPTERS

The book consists of thirteen chapters and two appendixes. Chapters 1 and 2 introduce basic concepts and properties of vectors and matrices, respectively. For instance, vector and matrix operations, norms, and inner products are introduced. Complex vectors and matrices are briefly discussed. This book mainly focuses on real vectors and matrices, though most of the results are valid for complex variables also. Even in such introductory chapters, we benefit from this exercise-solution-style textbook. For example, the proof of the Cauchy–Schwarz inequality (Exercise 1.9) is split into two questions (about a small lemma for the inner product and the proof of the inequality), so that students can take a step by step process. Another typical example is a question about a generalization of $x^2 = 0 \Leftrightarrow x = 0$ (Exercise 2.13):

For real matrices A, B and C show that:

(a) $A'A = O$ if and only if $A = O$;
(b) $AB = O$ if and only if $A'AB = O$;

(c) $AB = AC$ if and only if $A'AB = A'AC$.
(d) Why do we require the matrices to be real?

From this simple example, we can see several differences between this book and theorem-proof-style textbooks. First, in this exercise, (a) is used to show (b), and (b) is used to show (c). Thus, students can gradually proceed to more general and advanced problems. Second, by working out this exercise instead of just reading the proof, students can learn how to apply simpler results to more general problems. This skill of application is more important than knowing (or memorizing) the statements of theorems. Third, (d) is treated as a remark in theorem-proof-style textbooks. By making (d) a question, this book succeeds in drawing more attention from readers and stimulating their interest.

Chapter 3 is on vector spaces. Complex and real vector, L_2, inner-product, and Hilbert spaces are introduced, and related geometric concepts, such as subspaces, linear dependence, dimension, distance, and orthogonality, are discussed. Because other chapters focus on algebraic aspects, this chapter seems to be a small (but necessary) digression. Some exercises are a bit too advanced at this stage, and students might feel less motivated to learn about these abstract materials. So, I think Sections 3.2 (inner-product space) and 3.3 (Hilbert space) may as well be skipped or treated like appendixes.

Serious discussions take place from Chapter 4 on. This chapter deals with the rank, inverse, and determinant, which can be the first humps in learning matrix algebra. Each topic basically starts by a simple (and often numerical) example and then moves forward to a general result. For instance, in an exercise for expansions of the determinant by row and column vectors (Exercise 4.36), we first see an exercise on 3×3 matrix and then work on a general proof.

Chapter 5 extends the previous exercises to partitioned matrices. The inverse formula is provided as a main result, and several extensions of the results in Chapter 4 are presented. Because partitioned matrices appear frequently in econometrics, this separate and sufficient treatment on partitioned matrices is a distinguished feature of this book.

Chapter 6 is a collection of results and techniques for analyzing linear simultaneous equations. After introducing elementary and echelon matrices, techniques for solving equations (e.g., the Gaussian elimination and Cramér's rule) are provided. Exercises on (non)homogeneous equations follow. This chapter has several numerical exercises. The worked-exercise style of this book puts the same emphasis on numerical exercises as on theoretical ones. I found this very effective for learning; otherwise students might be tempted to skip numerical exercises.

Chapter 7, the longest chapter in this book, is on eigenvalues, eigenvectors, symmetric and triangular matrices, Schur's decomposition theorem, and Jordan decomposition and chains. Section 7.5 (Jordan's decomposition theorem) remarkably demonstrates the strength of this worked-exercise textbook. After three exercises on the Jordan blocks (Exercises 7.74–7.76), we prove a lemma for the Jordan's decomposition theorem in the cases of 2×2 and 3×3 matrices (Exercise 7.77) and $n \times n$ matrices (Exercise 7.78). Then, using these results, we tackle the Jordan's decomposition theorem (Exercise 7.79). Finally, we see three additional exercises on the theorem (an example of a Jordan matrix, a question on the number of Jordan blocks, and an alternative proof of the Cayley–Hamilton theorem). These step by step exercises effectively lead students to the high-end theorem. To catch the flavor of this book, I recommend that you read this section.

Chapter 8 deals with positive (semi)definite and idempotent matrices. Sections 8.1 and 8.2 provide implications of positive (semi)definiteness and decomposition theorems. Section 8.3 is a separate section on idempotent matrices, which appear frequently in regression analysis.

Chapter 9 is dedicated to matrix functions. Exercises on various series expansions for simple functions (e.g., geometric, logarithmic, and binomial expansions) and general representation theorems (Jordan and matrix-polynomial representation) are provided.

Chapter 10 is on the Kronecker product, vec operator, Moore–Penrose inverse, applications to linear equations, and generalized inverse. Each section is relatively short but is sufficient for first-year graduate courses. For a more extensive review of the Moore–Penrose and generalized inverse, the reader may refer to Rao and Mitra (1972).

Chapter 11 treats patterned matrices. Exercises on the commutation, symmetrizer, and duplication matrices are provided. Statistical examples using the Normal and Wishart distributions are quite helpful. A brief introduction to linear structures follows.

Chapter 12 is unique and useful. This chapter covers various matrix inequalities. As far as I know, standard textbooks on matrix algebra do not give a chapter on this topic. In this chapter, seven general principles for deriving matrix inequalities are first introduced. Then, they are succeeded by the Cauchy–Schwarz inequalities, positive (semi)definite matrix inequalities including Minkowski's, Kantorovich's, and Hadamard's inequalities, inequalities derived from the Schur complement, and inequalities concerning eigenvalues. After the student works them out, this chapter will be a handy toolbox for matrix inequalities.

Chapter 13 is on matrix (differential) calculus. This chapter has several statistical examples and might be considered a practical version of Magnus and Neudecker (1999). After a review on basic properties of differentials, Sections 13.2–13.7 are dedicated to differentials of various functions, such as scalar, vector, matrix, inverse, and determinant functions. Sections 13.8 and 13.10 deal with Jacobians and Hessians. Section 13.9 derives several sensitivities of linear regression estimators. Section 13.11 is on the Gauss–Markov theorem and multicollinearity. Section 13.12 analyzes the maximum likelihood estimator under the normal distribution. This chapter ends with exercises on matrix inequalities and equalities based on matrix calculus.

This book has two appendixes on some basic mathematical tools and notation. As the authors mention, the notation is based on a standard proposed by Abadir and Magnus (2002), and the other volumes of the series follow the same standard.

3. CONCLUSION

As the reader may have inferred from this review, I liked this book very much. This book reorganizes essential materials in matrix algebra from econometricians' perspectives and succeeds in providing students with techniques to solve numerical problems and prove theorems. Moreover, this book effectively trains students to develop new results by applying the existing ones. This book can be used either as a main or a supplementary text. Abadir and Magnus have done a valuable service to econometrics by writing this novel textbook. If the following volumes keep the same style and quality, the Econometric Exercises series will become a new standard for learning and teaching econometrics.

REFERENCES

Abadir, K.M. & J.R. Magnus (2002) Notation in econometrics: A proposal for a standard. *Econometrics Journal* 5, 76–90.

Harville, D.A. (1997) *Matrix Algebra from a Statistician's Perspective*. Springer-Verlag.

Horn, R.A. & C.R. Johnson (1990) *Matrix Analysis*. Cambridge University Press.

Lütkepohl, H. (1996) *Handbook of Matrices*. Wiley.

Magnus, J.R. & H. Neudecker (1999) *Matrix Differential Calculus with Applications in Statistics and Econometrics*, rev. ed. Wiley.

Phillips, P.C.B. & M.R. Wickens (1978) *Exercises in Econometrics*. Ballinger Publishing.

Rao, C.R. & S.K. Mitra (1972) *Generalized Inverse of Matrices and Its Applications*. Wiley.