The central limit theorem for Student’s distribution

Problem 03.6.1

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PROBLEMS AND SOLUTIONS

PROBLEMS

03.6.1. The Central Limit Theorem for Student’s Distribution
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Let \( x_1, \ldots, x_n \) be a random sample from Student’s \( t(\nu) \) distribution, where \( \nu \in \mathbb{R}_+ \). Investigate whether \( z_n := \sum_{i=1}^n x_i / \lambda_n \) is asymptotically \( N(0,1) \) for a suitable choice of \( \lambda_n \).

03.6.2. Unbiasedness of the OLS Estimator with Random Regressors
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Consider the linear regression model
\[ y = X\beta + u, \]
where \( X \) is an \( n \times k \) matrix of random regressors, \( u \) is an \( n \)-vector of error terms, and \( \beta \) is a \( k \)-vector of parameters. Suppose \( X \) has full column rank with probability one. It is a standard textbook claim that the ordinary least squares (OLS) estimator \( \hat{\beta} = (X'X)^{-1}X'y \) of \( \beta \) is unbiased if \( E(u|X) \overset{a.s.}{=} 0 \), where \( a.s. \) signifies almost sure equality. Specifically, it is claimed that unbiasedness follows from the law of iterated expectations and the relation \( E(\hat{\beta}|X) \overset{a.s.}{=} \beta + (X'X)^{-1}X'E(u|X). \) As it turns out, this argument is flawed.

(a) Show by example that \( E(u|X) \overset{a.s.}{=} 0 \) does not imply existence of \( E(\hat{\beta}) \).
(b) Provide stronger conditions under which \( E(\hat{\beta}) \) exists (and equals \( \beta \)).

SOLUTIONS

02.6.1. Oblique Projectors\(^1\) — Solution
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It is well known that an oblique projector \( P \) can be written as
\[ P = U \begin{pmatrix} I_r & K \\ 0 & 0 \end{pmatrix} U^*, \]
Solution

03.6.1 The Central Limit Theorem for Student’s Distribution—Solution

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Consider the Lindeberg–Feller central limit theorem (CLT), which we state as follows. Let \( \{x_n\} \) be a sequence of independent random variables with means \( \{\mu_n\} \) and nonzero variances \( \{\sigma_n^2\} \) (both existing), and c.d.f.s \( \{F_n\} \). Define \( \lambda_n > 0 \) by \( \lambda_n^2 = \sum_{i=1}^{n} \sigma_i^2 \). Then, Lindeberg’s condition

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \int_{|u-\mu_i|\geq \lambda_n \epsilon} \left( \frac{u-\mu_i}{\lambda_n} \right)^2 dF_i(u) = 0, \quad \forall \epsilon > 0,
\]

is equivalent to

\[
z_n := \frac{\sum_{i=1}^{n} (x_i - \mu_i)}{\lambda_n} \overset{d}{\sim} N(0,1) \quad \text{and} \quad \lim_{n \to \infty} \max_{1 \leq i \leq n} \Pr \left( \left| \frac{x_i - \mu_i}{\lambda_n} \right| \geq \epsilon \right) = 0,
\]

where the latter limit is known as the uniform asymptotic negligibility (u.a.n.) condition. One can usually interpret \( \lambda_n^2 \) as the variance of the numerator of \( z_n \). We shall see, however, that there are cases where asymptotic normality holds in spite of \( \{x_n\} \) having infinite variances.

Let \( \{x_n\} \) be a random sample from Student’s t(\( \nu \)). For \( \nu < 2 \), no \( \lambda_n \) exists that can lead to \( z_n := \sum_{i=1}^{n} x_i / \lambda_n \overset{d}{\sim} N(0,1) \). This is because the tails of the density of t(\( \nu \)) decay at a rate of \( u^{-\nu-1} \) and the stable limit theorem tells us that a nonnormal stable law arises if the tails of the p.d.f. of \( x_i \) decay at a rate of \( u^{-a} \) where \( a < 3 \); e.g., see Loève (1977, §25) or Hoffmann-Jørgensen (1994, §5.25). For example, for \( \nu = 1 \), the average of standard Cauchy variates is standard Cauchy too, so that there exists no \( \lambda_n \) achieving asymptotic normality of \( z_n \).

For \( \nu > 2 \), both the mean and the variance exist, and the Lindeberg–Feller CLT applies, with \( \lambda_n^2 = n \text{var}(x_i) \). The interesting part is \( \nu = 2 \), where we will show that asymptotic normality of \( z_n \) holds, in spite of \( \text{var}(x_i) \) being infinite, and we will derive the appropriate \( \lambda_n \). We will require the additional assump-

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tion that $\lambda_n^2 \to \infty$ as $n \to \infty$. In the standard CLT, this assumption was unnecessary, as it followed from $\lambda_n^2 = n \text{var}(x_i)$. We will see subsequently that $\lambda_n^2$ can be interpreted in terms of truncated variances for $\nu = 2$.

To prove the asymptotic normality of $z_n$, we need to show that the characteristic function $\varphi(t) := E(\exp(itx_i))$ satisfies

$$\lim_{n \to \infty} n \log \left( \frac{t}{\lambda_n} \right) = -\frac{t^2}{2}$$

for some choice of $\lambda_n$, with $\lambda_n \to \infty$ as $n \to \infty$. Because the sequence $\{x_n\}$ is i.i.d., the uniform asymptotic negligibility condition

$$\lim_{n \to \infty} \max_{1 \leq i \leq n} \Pr\left( \frac{|x_i|}{\lambda_n} \geq \epsilon \right) = 0$$

is satisfied for all $\epsilon > 0$, thus implying

$$\lim_{n \to \infty} \left| \varphi\left( \frac{t}{\lambda_n} \right) - 1 \right| = 0.$$

This allows us to take the leading term of the logarithmic expansion of the left-hand side of (1) as

$$\lim_{n \to \infty} n \log \varphi\left( \frac{t}{\lambda_n} \right) = \lim_{n \to \infty} n \left( \varphi\left( \frac{t}{\lambda_n} \right) - 1 \right)$$

$$= -\frac{t^2}{2} \lim_{n \to \infty} n \int_{|u| < \lambda_n \epsilon} \frac{u^2}{\lambda_n^2} \text{d}F(u), \quad \lambda_n \to \infty,$$

where the linear term in $t$ drops out because the sequence $\{x_n\}$ is centered around zero. Asymptotic standard-normality obtains if we can find the appropriate $\lambda_n^2$ that makes the latter limit equal to 1 for all $\epsilon > 0$. Notice that this limit is the complement of Lindeberg’s condition, where $\sum_{i=1}^{n}$ is replaced by $n$ because $\{x_n\}$ is an i.i.d. sequence.

From Student’s $t(2)$ density,

$$\int_{-\sqrt{2}}^{\sqrt{2}} \frac{u^2}{\sqrt{8} \left( 1 + \frac{u^2}{2} \right)^{3/2}} \text{d}u = 2 \log(\sqrt{1 + c^2} + c) - \frac{2c}{\sqrt{1 + c^2}}$$

$$= 2 \sinh^{-1}(c) - \frac{2c}{\sqrt{1 + c^2}}$$
tends to infinity as $c \rightarrow \infty$. We need to solve

$$1 = \lim_{n \to \infty} \frac{n}{\lambda_n^2} \int_{|u| < \lambda_n e} u^2 \, dF(u) = \lim_{n \to \infty} \frac{2n \sinh^{-1}(\lambda_n e/\sqrt{2})}{\lambda_n^2}$$

where we have dropped $2c/\sqrt{1+c^2} \rightarrow 2$ that is dominated by $\sinh^{-1}(c) \rightarrow \infty$. By using the logarithmic representation of the latter and simplifying,

$$1 = \lim_{n \to \infty} \frac{2n \log(\lambda_n)}{\lambda_n^2}$$

is solved by $\lambda_n = \sqrt{n \log(n)}$ or any other function that is asymptotically equivalent to it (such as $\sqrt{n \log(n) + \sqrt{n}}$). Therefore,

$$z_n = \frac{1}{\sqrt{n \log(n)}} \sum_{i=1}^n x_i \overset{a}{\sim} N(0,1).$$

**REFERENCES**


**03.6.2. Unbiasedness of the OLS Estimator with Random Regressors—Solution**

Michael Jansson (theposer of the problem)

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(a) Suppose $n = 1$ and let $X$ and $u$ be independent standard normal variates. Then $X$ is nonzero with probability one and $E(u|X) \overset{a.s.}{=} 0$, but $E(\hat{\beta}) = \infty$ because the distribution of $\hat{\beta} - \beta = u/X$ is Cauchy.

(b) The matrix $X$ has full column rank with probability one if and only if

$$\Pr[\lambda_{\min}(X'X) > 0] = 1,$$

where $\lambda_{\min}(\cdot)$ denotes the minimal eigenvalue of the argument.

$E(\hat{\beta})$ exists if (and only if) $E[c'(\hat{\beta} - \beta)] < \infty$ for any $k$-vector $c$ with $c'c = 1$. In the sequel, let $c$ be an arbitrary $k$-vector with unit length. Now,

$$|c'(\hat{\beta} - \beta)| = |c'(X'X)^{-1}X'u| \leq \sqrt{c'(X'X)^{-1}c\sqrt{u'u}} \leq \sqrt{\lambda_{\min}^{-1}(X'X)} \sqrt{u'u},$$

where the first inequality uses the Cauchy–Schwarz inequality and the second inequality uses Magnus and Neudecker (1988), Theorem 11.4.

If $X$ and $u$ are independent, $E(u|X) \overset{a.s.}{=} 0$, and

$$E[\lambda_{\min}^{-1/2}(X'X)] < \infty,$$

(3)